Cell-centered discretization methods for 2nd order elliptic problems

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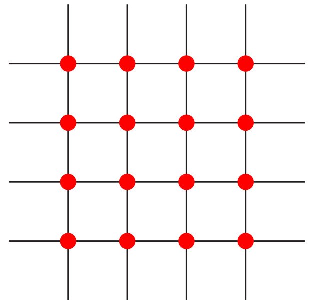
INTRODUCTION

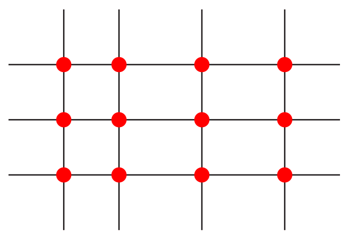
Vertex-centered approximation methods

The degrees of freedom are located at the vertices of the mesh

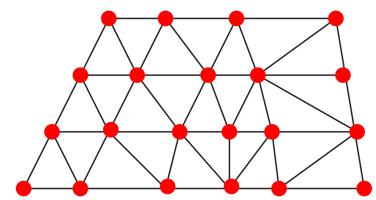
Finite differences



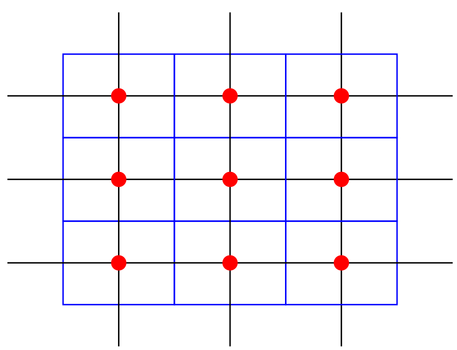
Finite elements Q_1

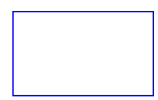


P_1



Vertex-centered finite volumes

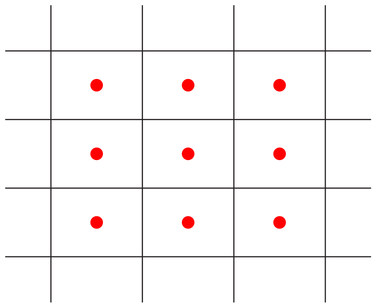




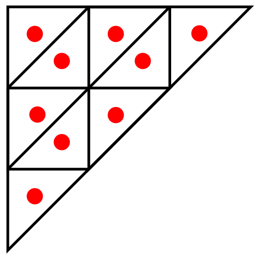
control volume

Cell-centered approximation methods

Mixed finite elements

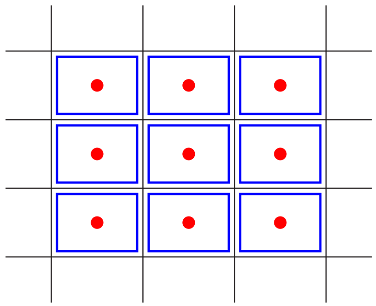


on rectangles



on triangles

Finite volumes

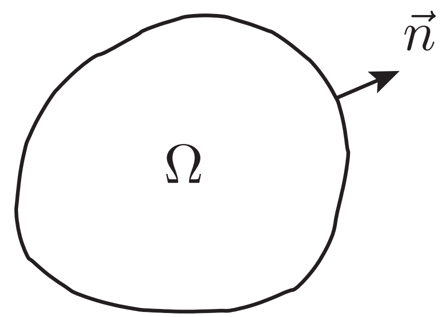


Unknowns : average value in each cell   Control volume = cell

Nodal methods in neutronics

The model problem

$$
\begin{aligned}
\text{div}(-K\, \vec{\text{grad}}\; p) &= f \quad \text{in } \Omega \\
p &= \bar{p} \quad \text{on } \partial\Omega \quad \text{if Dirichlet} \\
-K\frac{\partial p}{\partial n} &= g \quad \text{on } \partial\Omega \quad \text{if Neumann}
\end{aligned}
$$



For flow in porous media :

p, pressure K, permeability $\vec{u} = -K\, \vec{\text{grad}}\; p$, Darcy velocity

$$
K = \begin{bmatrix} k^1 & k^{12} \\ k^{12} & k^2 \end{bmatrix}, \quad 0 < \alpha \le (Kx, x) \le \beta, x \in \mathbb{R}^2.
$$

The Sobolev space $H^1(\Omega)$.

$$H^0(\Omega) = L^2(\Omega) \qquad \|q\|_{0,\Omega}^2 = \int_\Omega q^2(x)dx$$

$$H^1(\Omega) = \{q \in L^2(\Omega);\ \vec{\text{grad}}\ q \in (L^2(\Omega))^2\}$$

$$\|q\|_{1,\Omega}^2 = \|q\|_{0,\Omega}^2 + |q|_{1,\Omega}^2 \qquad |q|_{1,\Omega}^2 = \int_\Omega |\ \vec{\text{grad}}\ q|^2(x)dx$$

The trace $q|_\Gamma$ of $q \in H^1(\Omega)$ is in $H^{1/2}(\Gamma)$.

The trace $\dfrac{\partial q}{\partial n}|_\Gamma$ of $q \in H^1(\Omega)$ is in $H^{-1/2}(\Gamma)$, the dual space of $H^{1/2}(\Gamma)$.

Weak primal formulation

Assume $K \in L^\infty(\Omega), f \in L^2(\Omega)$.

- Neumann boundary conditions: $g \in H^{-1/2}(\partial\Omega)$.

Find $p \in H^1(\Omega)$ such that

$$\int_\Omega K \, \vec{\text{grad}} \, p \cdot \, \vec{\text{grad}} \, q = \int_\Omega f q - < g, q >, \quad q \in H^1(\Omega).$$

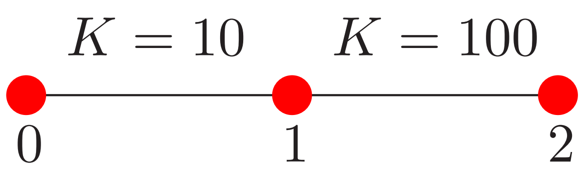
- Dirichlet boundary conditions: $\overline{p} \in H^{1/2}(\partial\Omega)$.

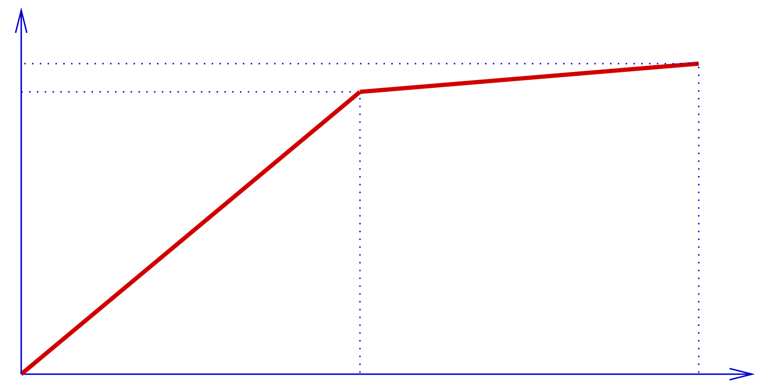
Find $p \in V_{\overline{p}} = \{q \in H^1(\Omega), q = \overline{p} \text{ on } \partial\Omega\}$ such that

$$\int_\Omega K \, \vec{\text{grad}} \, p \cdot \, \vec{\text{grad}} \, q = \int_\Omega f q, \quad q \in V_0.$$

An example with a discontinuous K

In one dimension, $\Omega =]0,2[,\ f = 0,\ p(0) = 0,\ p(2) = 1.$





$$\frac{\partial u}{\partial x} = 0 \quad \Longrightarrow u \text{ constant thus very smooth.}$$

p continuous, piecewise linear, $\dfrac{\partial p}{\partial x}$ discontinuous at $x = 1 \Longrightarrow p$ is not smooth.

u has a physical meaning and is a good mathematical and numerical unknown.

APPROXIMATION WITH MIXED FINITE ELEMENTS

Mixed formulation

Write the elliptic problem as a system of first order equations:

$$\text{div}\,\vec{u} = f, \quad \vec{u} = -K\ \vec{\text{grad}}\ p, \ \text{in } \Omega$$

$$p = \overline{p} \text{ on } \Gamma_D, \quad \vec{u}\cdot\vec{n} = g \text{ on } \Gamma_N, \quad \Gamma_N \cup \Gamma_D = \Gamma = \partial\Omega.$$

Assume that $f \in L^2(\Omega)$ so $\text{div}\vec{u} \in L^2(\Omega)$.
Therefore we take $\vec{u} \in H(\text{div},\Omega) = \{v \in (L^2(\Omega))^2; \text{div}\ \vec{v} \in L^2(\Omega)\}$.

Multiply the second equation by K^{-1}, then by $\vec{v}$, integrate over Ω and by parts.
We obtain $\displaystyle\int_\Omega (K^{-1}\vec{u})\cdot\vec{v} - \int_\Omega p\text{div}\vec{v} = - < p, \vec{v}\cdot\vec{n} >$

Recall Green's formula: $\displaystyle\int_\Omega \vec{\text{grad}}\ q\cdot\vec{v} + \int_\Omega q\,\text{div}\vec{v} = \int_\Gamma q\vec{v}\cdot\vec{n}.$

It is sufficient to take $p \in L^2(\Omega), \vec{u} \in H(\text{div},\Omega)$.

Properties of $H(\textbf{div}, \Omega)$

$H(\text{div}, \Omega) = \{v \in (L^2(\Omega))^2; \text{div}\ \vec{v} \in L^2(\Omega)\}$ is an Hilbert space with norm

$$\|\vec{v}\|_{H(\text{div},\Omega)} = \|\vec{v}\|_{L^2(\Omega)} + \|\text{div}\vec{v}\|_{L^2(\Omega)}.$$

Traces $(\vec{v} \cdot \vec{n})|_\Gamma$ of functions $\vec{v}$ of $H(\text{div}, \Omega)$ are in $H^{-1/2}(\Gamma)$,

so boundary data must be such that $\overline{p} \in H^{1/2}(\Gamma_D), \quad g \in H^{-1/2}(\Gamma_N)$.

Notations

The data are $f \in L^2(\Omega), \overline{p} \in H^{1/2}(\Gamma_D), g \in H^{-1/2}(\Gamma_N)$.

The spaces are $\mathcal{M} = L^2(\Omega),\ \mathcal{W} = H(\text{div}, \Omega),\ \mathcal{W}_g = \{\vec{v} \in \mathcal{W}; \vec{v}\cdot\vec{n} = g \text{ on } \Gamma_N\}$.

Introduce the forms

$$
\begin{array}{llllll}
a: & (L^2(\Omega))^2 \times (L^2(\Omega))^2 & \longrightarrow \mathbb{R}, & a(\vec{u}, \vec{v}) & = & \displaystyle\int_\Omega (K^{-1}\vec{u}) \cdot \vec{v}, \\
b: & \mathcal{W} \times \mathcal{M} & \longrightarrow \mathbb{R}, & b(\vec{v}, q) & = & \displaystyle\int_\Omega q \, \text{div}\vec{v}, \\
l_{\mathcal{W}}: & \mathcal{W} & \longrightarrow \mathbb{R}, & l_{\mathcal{W}}(\vec{v}) & = & \displaystyle\int_{\Gamma_D} -\overline{p}\, \vec{v} \cdot \vec{n}, \\
l_{\mathcal{M}}: & \mathcal{M} & \longrightarrow \mathbb{R}, & l_{\mathcal{M}}(\vec{v}) & = & \displaystyle\int_\Omega f q.
\end{array}
$$

Mixed formulation

The problem is

$$\mathcal{P}_m \left\{ \begin{array}{llll} \multicolumn{4}{l}{\text{Find } \vec{u} \in \mathcal{W}_g \text{ and } p \in \mathcal{M} \text{ such that}} \\ a(\vec{u},\vec{v}) \quad -b(\vec{v},p) & = & l_{\mathcal{W}}(\vec{v}), & \vec{v} \in \mathcal{W}_0, \\ b(\vec{u},q) & = & l_{\mathcal{M}}(q), & q \in \mathcal{M}. \end{array} \right.$$

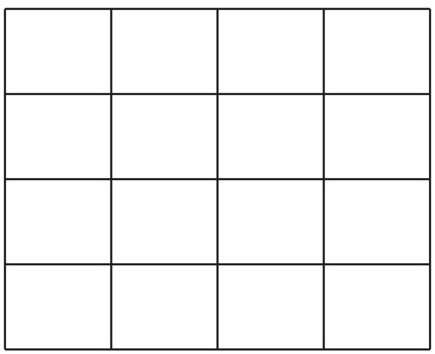
Discretization of the domain

1. Let $\mathcal{T}_h$ be a discretization of Ω. $\quad$ $\text{Card}(\mathcal{T}_h) = ne =$ number of cells.

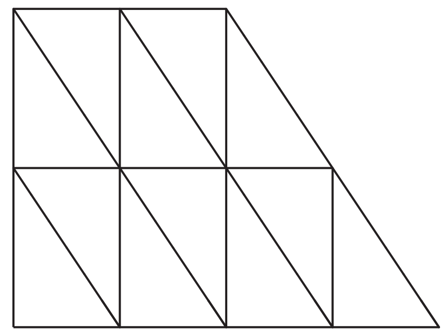
   $\mathcal{A}_h$ be the set of edges. $\quad$ $\text{Card}(\mathcal{A}_h) = na =$ number of edges

   h the largest diameter of the cells.

a rectangular mesh



a triangular mesh



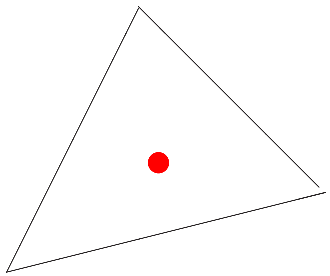
2. Define

- an approximation p_h of p in $\mathcal{M}_h$, a finite dimensional subset of $\mathcal{M}$
- an approximation $\vec{u}_h$ of $\vec{u}$ in $\mathcal{W}_h$, a finite dimensional subset of $\mathcal{W}$.

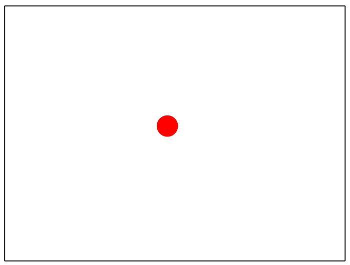
Approximation space for the scalar unknown

$\mathcal{M}_h$ = the space of functions q_h in $\mathcal{M}$ which are

constant over each triangle



constant over each rectangle



dim $\mathcal{M}_h = ne$ = number of elements

The degrees of freedom are
p_T an approximation of the average value of p over the cell T, $\quad T \in \mathcal{T}_h$.

A basis is $\{\chi_T\}_{T\in\mathcal{T}_h}$ such that $\chi_T(x) = \begin{cases} 1 & \text{if } x \in T \\ 0 & \text{otherwise} \end{cases}$

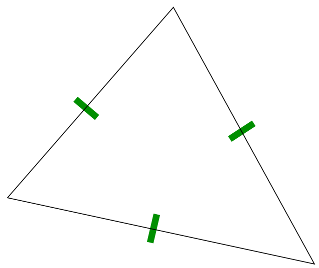
Then $p_h = \sum_{T\in\mathcal{T}_h} p_T \chi_T$

Approximation of the vector unknown

$\mathcal{W}_h = \{\vec{v}_h \in \mathcal{W}; \vec{v}_h|T \in \mathcal{W}_T, T \in \mathcal{T}_h\}.$

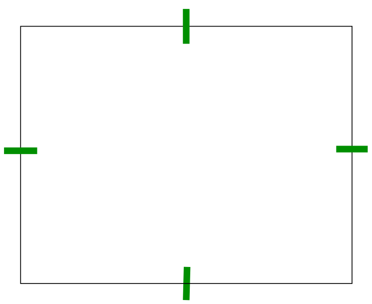
on each triangle

$$\mathcal{W}_T = \{\vec{v}_h; \vec{v}_{h|T} = a \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} b \\ c \end{pmatrix}\}$$



on each rectangle

$$\mathcal{W}_T = \{\vec{v}_h; \vec{v}_{h|T} = \begin{pmatrix} ax_1 + b \\ cx_2 + d \end{pmatrix}\}$$



Functions $\vec{v} \in \mathcal{W}_T$ are uniquely defined by $\int_E \vec{v} \cdot \vec{n}_T, E \subset \partial T$.

On remarque que div$\mathcal{W}_h = \mathcal{M}_h$.

dim$\mathcal{W}_h = na =$ number of edges.

The degrees of freedom for $\mathcal{W}_h$ are

u_E an approximation of the flow rate of $\vec{u}$ across E, $\int_E \vec{u} \cdot \vec{n}_E,\ E \in \mathcal{A}_h$,

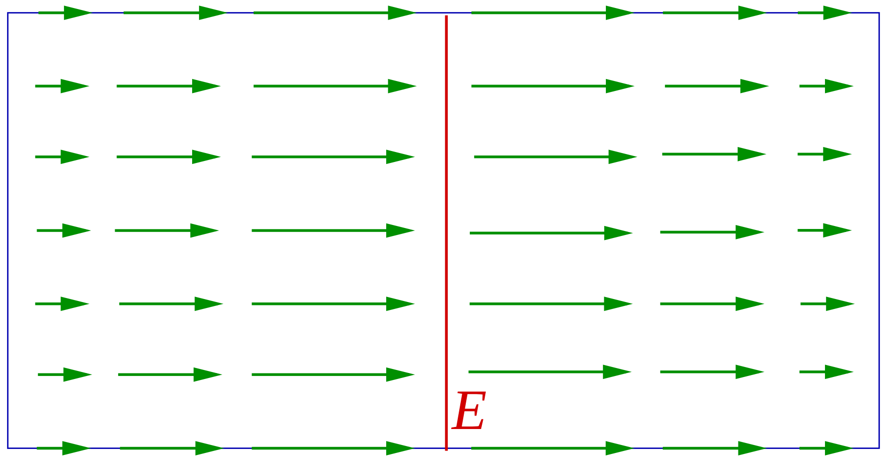
$\vec{n}_E$ a chosen unit normal to E.

A basis of $\mathcal{W}_h$ is $\{\vec{v}_E\}_{E\in\mathcal{A}_h}$ such that $\int_F \vec{v}_E \cdot \vec{n}_F = \delta_{E,F}, F \in \mathcal{A}_h$.

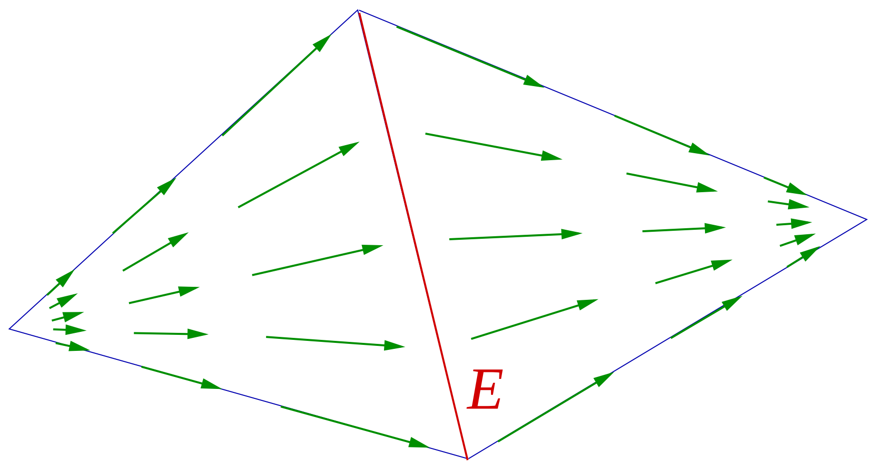
Then, $\vec{u}_h = \sum_{E\in\mathcal{A}_h} u_E \vec{v}_E$.

Basis functions of $\mathcal{W}_h$

- For rectangles $\vec{v}_E$ is:



- For triangles $\vec{v}_E$ is:



The approximation problem

Assume the data $\overline{p}$, g are piecewise constant on the edges $E \subset \Gamma$.

Introduce $\mathcal{W}_{hg} = \{\vec{v} \in \mathcal{W}_h; \vec{v} = g \text{ on } \Gamma_N\}$

The approximation problem is

$$\mathcal{P}_{mh} \left\{ \begin{array}{llcll} \multicolumn{5}{l}{\text{Find } \vec{u}_h \in \mathcal{W}_{hg} \text{ and } p_h \in \mathcal{M}_h \text{ such that}} \\ a(\vec{u}_h, \vec{v}_h) & -b(\vec{v}_h, p_h) & = & l_{\mathcal{W}}(\vec{v}_h), & \vec{v}_h \in \mathcal{W}_{h0}, \\ b(\vec{u}_h, q) & & = & l_{\mathcal{M}}(q_h), & q_h \in \mathcal{M}_h. \end{array} \right.$$

Discrete equations

The unknowns are the degrees of freedom:

Find $\{p_T\}_{T\in\mathcal{T}_h}, \{u_E\}_{E\in\mathcal{A}_h}$ such that

$$\int_\Omega K^{-1} \sum_{F\in\mathcal{A}_h} u_F \vec{v}_F \cdot \vec{v}_E - \int_\Omega \sum_{T\in\mathcal{T}_h} p_T \chi_T \,\mathrm{div}\, \vec{v}_E = \int_{\Gamma_D} -\overline{p}\, \vec{v}_E \cdot \vec{n}, \quad E \in \mathcal{A}_h, E \not\subset \Gamma_N$$

$$\int_\Omega \mathrm{div} \sum_{E\in\mathcal{A}_h} u_E \vec{v}_E\, \chi_T = \int_\Omega f\, \chi_T, \qquad T \in \mathcal{T}_h$$

$u_E = g|E|, \quad E \subset \Gamma_N$ (assuming $\vec{n}_E = \vec{n}$)

Find $\{p_T\}_{T\in\mathcal{T}_h}, \{u_E\}_{E\in\mathcal{A}_h}$ such that

$$\sum_{F\in\mathcal{A}_h} u_F \int_\Omega K^{-1} \vec{v}_F \cdot \vec{v}_E - \sum_{T\in\mathcal{T}_h} p_T \int_\Omega \chi_T \,\mathrm{div}\, \vec{v}_E = \int_{\Gamma_D} -\overline{p}\, \vec{v}_E \cdot \vec{n}, \quad E \in \mathcal{A}_h, E \not\subset \Gamma_N$$

$$\sum_{E\in\mathcal{A}_h} u_E \int_\Omega \mathrm{div} \vec{v}_E\, \chi_T = \int_\Omega f\, \chi_T, \qquad T \in \mathcal{T}_h$$

$u_E = g|E|, \quad E \subset \Gamma_N.$

Algebraic system

This leads to the linear system

$$\begin{bmatrix} A & -{}^tD \\ D & 0 \end{bmatrix} \begin{bmatrix} U \\ P \end{bmatrix} = \begin{bmatrix} F_v \\ F_q \end{bmatrix}$$

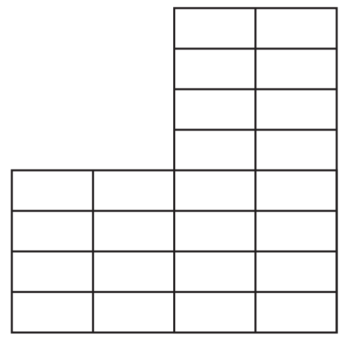
with $P = \{p_T\}_{T \in \mathcal{T}_h}, U = \{u_E\}_{E \in \mathcal{A}_h, E \not\subset \Gamma_N}$.

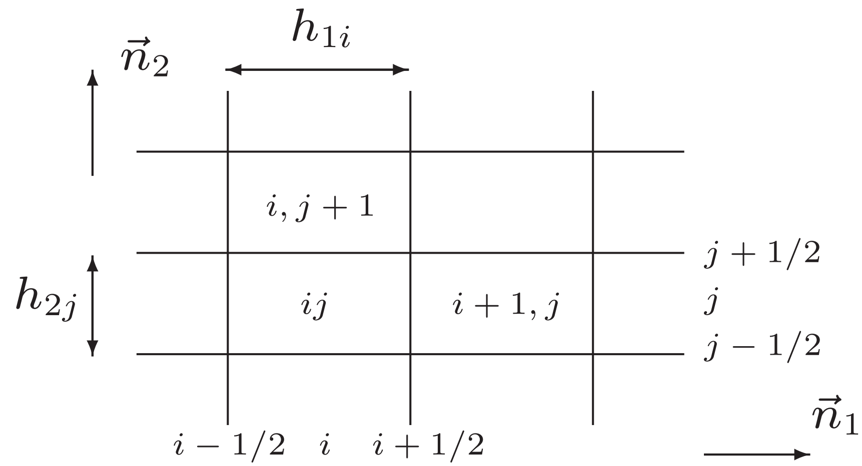
This linear system is not positive-definite.

- For triangles A has 5 nonzero entries per row
- For quadrilaterals A has 7 nonzero entries per row

On a rectangular mesh

A rectangular mesh for Ω





We take $\vec{n}_E = \vec{n}_1$ if E is vertical, $\vec{n}_E = \vec{n}_2$ if E is horizontal.

Note that

$$\sum_{E \in \mathcal{A}_h} u_E \int_\Omega \text{div} \vec{v}_E \, \chi_T = \sum_{E \subset \partial T} u_E \int_T \text{div} \vec{v}_E.$$

Thus the second discrete equation gives

$$u_{i+1/2,j} - u_{i,j-1/2} + u_{i,j+1/2} - u_{i-1/2,j} = \int_{T_{ij}} f$$

Consider now the first discrete equation.

Denote $\mathcal{N}(E)$ the set of the $\begin{cases} 2 & \text{cells adjacent to } E \text{ if } E \not\subset \Gamma \\ 1 & \text{cell adjacent to } E \text{ if } E \subset \Gamma \end{cases}$

- If $E = E_{i+1/2,j}, E \not\subset \Gamma_D$:

$$\sum_{T \in \mathcal{T}_h} p_T \int_\Omega \chi_T \operatorname{div} \vec{v}_E = \sum_{T \in \mathcal{N}(E)} p_T \int_T \operatorname{div} \vec{v}_E = p_{ij} - p_{i+1,j}.$$

- If $E = E_{i+1/2,j}, E \subset \Gamma_D$:

$$\sum_{T \in \mathcal{T}_h} p_T \int_\Omega \chi_T \operatorname{div} \vec{v}_E = p_{ij} \text{ assuming } E \text{ lies on the right of the domain,}$$

$$\sum_{T \in \mathcal{T}_h} p_T \int_\Omega \chi_T \operatorname{div} \vec{v}_E = -p_{i+1,j} \text{ assuming } E \text{ lies on the left of the domain.}$$

- If $E = E_{i+1/2,j}, E \not\subset \Gamma$:

$$\sum_{F \in \mathcal{A}_h} u_F \int_\Omega K^{-1}\vec{v}_F \cdot \vec{v}_E = \sum_{T \in \mathcal{N}(E)} \sum_{F \in \partial T} u_F \int_\Omega K^{-1}\vec{v}_F \cdot \vec{v}_E =$$

$$u_{i+1/2,j} \int_{T_{ij}} K^{-1}\vec{v}_{i+1/2,j} \cdot \vec{v}_{i+1/2,j} + u_{i-1/2,j} \int_{T_{ij}} K^{-1}\vec{v}_{i-1/2,j} \cdot \vec{v}_{i+1/2,j} +$$

$$u_{i,j+1/2} \int_{T_{ij}} K^{-1}\vec{v}_{i,j+1/2} \cdot \vec{v}_{i+1/2,j} + u_{i,j-1/2} \int_{T_{ij}} K^{-1}\vec{v}_{i,j-1/2} \cdot \vec{v}_{i+1/2,j} +$$

$$u_{i+1/2,j} \int_{T_{i+1,j}} K^{-1}\vec{v}_{i+1/2,j} \cdot \vec{v}_{i+1/2,j} + u_{i+3/2,j} \int_{T_{i+1,j}} K^{-1}\vec{v}_{i+3/2,j} \cdot \vec{v}_{i+1/2,j} +$$

$$u_{i+1,j+1/2} \int_{T_{i+1,j}} K^{-1}\vec{v}_{i+1,j+1/2} \cdot \vec{v}_{i+1/2,j} + u_{i+1,j-1/2} \int_{T_{i+1,j}} K^{-1}\vec{v}_{i+1,j-1/2} \cdot \vec{v}_{i+1/2,j}$$

- If $E \subset \Gamma$, say for instance $i = 0$ (left boundary):

$$\sum_{F \in \mathcal{A}_h} u_F \int_\Omega K^{-1}\vec{v}_F \cdot \vec{v}_E = \sum_{T \in \mathcal{N}(E)} \sum_{F \in \partial T} u_F \int_\Omega K^{-1}\vec{v}_F \cdot \vec{v}_E =$$

$$u_{1/2,j} \int_{T_{1,j}} K^{-1}\vec{v}_{1/2,j} \cdot \vec{v}_{1/2,j} + u_{3/2,j} \int_{T_{1,j}} K^{-1}\vec{v}_{3/2,j} \cdot \vec{v}_{1/2,j} +$$

$$u_{1,j+1/2} \int_{T_{1,j}} K^{-1}\vec{v}_{1,j+1/2} \cdot \vec{v}_{1/2,j} + u_{1,j-1/2} \int_{T_{1,j}} K^{-1}\vec{v}_{1,j-1/2} \cdot \vec{v}_{1/2,j}$$

Denote $K^{-1} = \begin{bmatrix} \alpha^1 & \alpha^{12} \\ \alpha^{12} & \alpha^2 \end{bmatrix}$,

with $\alpha^1 = k^2/\kappa, \alpha^2 = k^1/\kappa, \alpha^{12} = -k^{12}/\kappa, \kappa = k^1 k^1 - (k^{12})^2$.

$$\int_{T_{ij}} K^{-1}\vec{v}_{i+1/2,j} \cdot \vec{v}_{i+1/2,j} = \frac{\alpha^1_{ij}}{3}\frac{h_{1i}}{h_{2j}}$$

$$\int_{T_{ij}} K^{-1}\vec{v}_{i-1/2,j} \cdot \vec{v}_{i+1/2,j} = \frac{\alpha^1_{ij}}{6}\frac{h_{1i}}{h_{2j}}$$

$$\int_{T_{ij}} K^{-1}\vec{v}_{i,j+1/2} \cdot \vec{v}_{i+1/2,j} = \frac{\alpha^{12}_{ij}}{4}$$

When K is diagonal

Products of basis functions for vertical edges by basis functions for horizontal edges vanish.

- If $E = E_{i+1/2,j}, E \not\subset \Gamma$:

$$\sum_{F \in \mathcal{A}_h} u_F \int_\Omega K^{-1} \vec{v}_F \cdot \vec{v}_E = \sum_{T \in \mathcal{N}(E)} \sum_{F \in \partial T} u_F \int_\Omega K^{-1} \vec{v}_F \cdot \vec{v}_E =$$

$$u_{i+1/2,j} \left[\int_{T_{ij}} \frac{1}{k^1_{ij}} \vec{v}_{i+1/2,j} \cdot \vec{v}_{i+1/2,j} + \int_{T_{i+1,j}} \frac{1}{k^1_{i,j+1}} \vec{v}_{i+1/2,j} \cdot \vec{v}_{i+1/2,j} \right] +$$

$$u_{i-1/2,j} \int_{T_{ij}} \frac{1}{k^1_{ij}} \vec{v}_{i-1/2,j} \cdot \vec{v}_{i+1/2,j} + u_{i+3/2,j} \int_{T_{i+1,j}} \frac{1}{k^1_{i,j+1}} \vec{v}_{i+3/2,j} \cdot \vec{v}_{i+1/2,j}$$

- If $E \subset \Gamma$, say for instance $i = 0$ (left boundary):

$$\sum_{F \in \mathcal{A}_h} u_F \int_\Omega K^{-1} \vec{v}_F \cdot \vec{v}_E = \sum_{T \in \mathcal{N}(E)} \sum_{F \in \partial T} u_F \int_\Omega K^{-1} \vec{v}_F \cdot \vec{v}_E =$$

$$u_{1/2,j} \int_{T_{1,j}} \frac{1}{k^1_{1j}} \vec{v}_{1/2,j} \cdot \vec{v}_{1/2,j} + u_{3/2,j} \int_{T_{1,j}} \frac{1}{k^1_{1,j+1}} \vec{v}_{3/2,j} \cdot \vec{v}_{1/2,j}$$

Using V, H as indices for the vertical and the horizontal edges we can write the linear system as

$$\begin{bmatrix} A_V & 0 & -{}^tD_V \\ 0 & A_H & -{}^tD_H \\ D_V & D_H & 0 \end{bmatrix} \begin{bmatrix} U_V \\ U_H \\ P \end{bmatrix} = \begin{bmatrix} F_{vV} \\ F_{vH} \\ F_q \end{bmatrix}$$

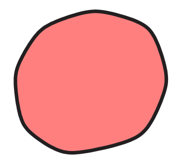
Matrices A_V et A_H are tridiagonal.

FINITE VOLUMES

The PDE as a conservation law.

$\text{div}\vec{u} = f$ is a conservation law.

This equation is obtained in two steps:

1. Write mass balance in a small volume T 

$$\underbrace{\int_{\partial T} \vec{u} \cdot \vec{n}}_{\text{échange}} = \underbrace{\int_T f}_{\text{exterior contribution}} \quad \begin{cases} > 0 \text{ source} \\ < 0 \text{ well} \\ = 0 \text{ what goes in goes out} \end{cases}$$

2. Make T infinitely small 

Finite volumes as a method which models the physics

In a finite volume method:

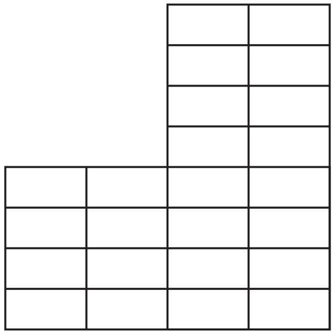
1. skip step 2
2. T is a triangle or a rectangle
3. p is constant over T
4. give a rule to calculate $\vec{u} \cdot \vec{n}$ on the edges

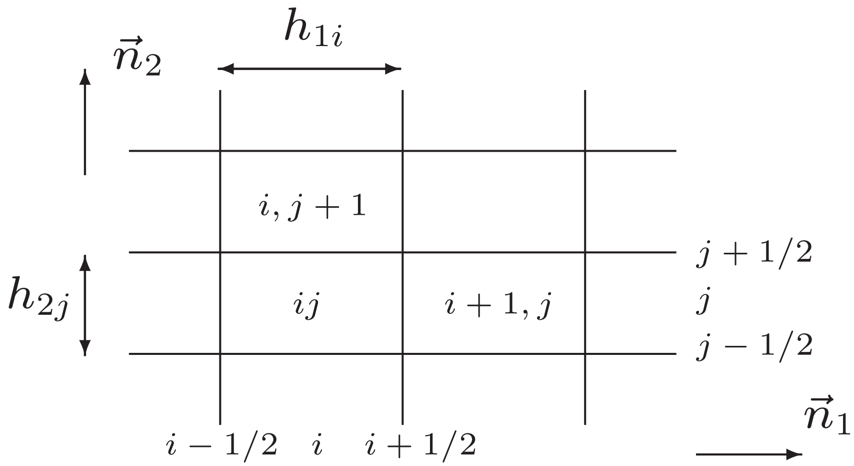
Classical formulation of cell-centered finite volumes

div$\vec{u} = f, \quad \vec{u} = -K \, \vec{\text{grad}} \, p$ dans Ω

$p = \overline{p}$ sur $\partial\Omega$

A rectangular mesh for Ω





The unknowns :

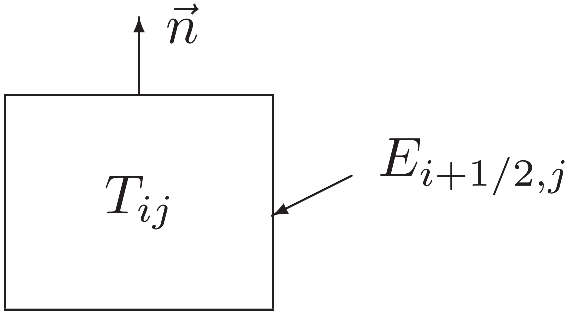
p_T an approximation of the average value of p over the cell T.

u_E an approximation of $\displaystyle\int_E \vec{u} \cdot \vec{n}_E$, $\quad \vec{n}_E = \vec{n}_1$ if E vertical

$\vec{n}_E = \vec{n}_2$ if E horizontal

Rectangular finite volumes

Integrate over the cell T_{ij} :



$$\int_{T_{ij}} \text{div}\vec{u} = \int_{\partial T_{ij}} \vec{u} \cdot \vec{n} = \int_{T_{ij}} f$$

$$\int_{E_{i,j-1/2}} \vec{u} \cdot \vec{n} + \int_{E_{i+1/2,j}} \vec{u} \cdot \vec{n} + \int_{E_{i,j+1/2}} \vec{u} \cdot \vec{n} + \int_{E_{i-1/2,j}} \vec{u} \cdot \vec{n} = \int_{T_{ij}} f$$

This equation is approximated by

$$u_{i+1/2,j} - u_{i,j-1/2} + u_{i,j+1/2} - u_{i-1/2,j} = \int_{T_{ij}} f$$

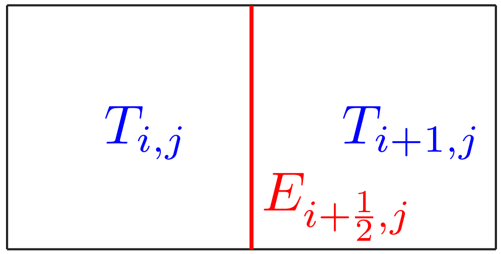
A finite volume consists in specifying

how to calculate the $u_{i+1/2,j}$'s in terms of the p_{ij}'s.

Assume K is diagonal: $K = \begin{bmatrix} k^1 & 0 \\ 0 & k^2 \end{bmatrix}$.

It is natural to approximate $\vec{u} = -K\ \vec{\text{grad}}\ p$

by $$u_{i+\frac{1}{2},j} = -k^1_{i+\frac{1}{2},j} \frac{p_{i+1,j} - p_{i,j}}{\frac{h_{1i}+h_{1,i+1}}{2}} h_{2j}$$

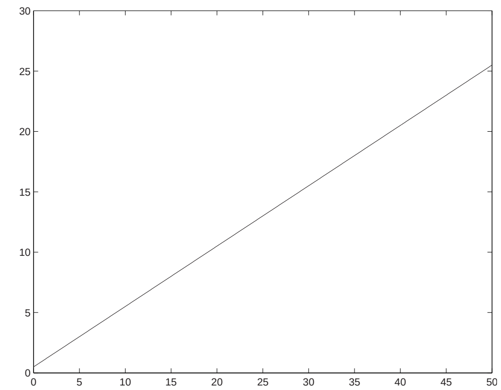


How to choose $k^1_{i+\frac{1}{2},j}$?

$$k^1_{i+\frac{1}{2},j} = \begin{cases} \frac{1}{2}(k^1_{i,j} + k^1_{i+1,j}) & \text{the arithmetic average} \\ \dfrac{1}{\frac{1}{2}(\frac{1}{k^1_{i,j}} + \frac{1}{k^1_{i+1,j}})} & \text{the harmonic average} \end{cases}$$

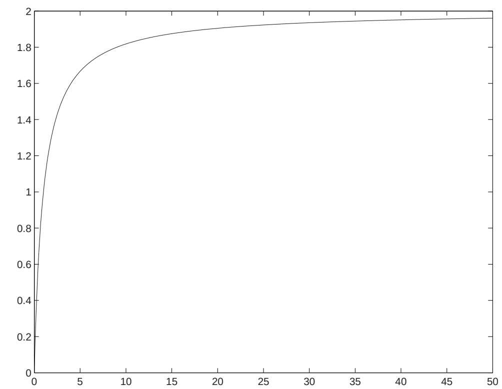
Arithmetic average: $\frac{1}{2}(s+r)$

For a fixed s: $\begin{cases} \xrightarrow{r \to \infty} & \infty \\ \xrightarrow{r \to 0} & \frac{s}{2} \end{cases}$



Harmonic average: $\frac{1}{\frac{1}{2}(\frac{1}{s}+\frac{1}{r})}$

For a fixed s: $\begin{cases} \xrightarrow{r \to \infty} & 2s \\ \xrightarrow{r \to 0} & 0 \end{cases}$



The harmonic average is better
since it allows no flow to enter a cell with 0 permeability.

Combining

$$u_{i+\frac{1}{2},j} - u_{i-\frac{1}{2},j} + u_{i,j+\frac{1}{2}} - u_{i,j-\frac{1}{2}} = \int_{T_{ij}} f$$

with

$$u_{i+\frac{1}{2},j} = -k^1_{i+\frac{1}{2},j} \, \frac{p_{i+1,j} - p_{i,j}}{\frac{h_{1,i+1}+h_{1i}}{2}} h_{2j}$$

we obtain

$$-k^1_{i+\frac{1}{2},j} \, \frac{p_{i+1,j} - p_{i,j}}{\frac{h_{1,i+1}+h_{1i}}{2}} h_{2j} + k^1_{i-\frac{1}{2},j} \, \frac{p_{ij} - p_{i-1,j}}{\frac{h_{1i}+h_{1,i-1}}{2}} h_{2j}$$

$$-k^2_{i,j+\frac{1}{2}} \, \frac{p_{i,j+1} - p_{i,j}}{\frac{h_{2,j+1}+h_{2j}}{2}} h_{1i} + k^2_{i,j-\frac{1}{2}} \, \frac{p_{ij} - p_{i-1,j}}{\frac{h_{2j}+h_{2,j-1}}{2}} h_{1i} = \int_{T_{ij}} f$$

We divide by $h_{1i}h_{2j}$ and that gives the five point scheme:

$$-p_{i+1,j}\frac{k^1_{i+\frac{1}{2},j}}{h_{1i}\frac{h_{1,i+1}+h_{1i}}{2}} - p_{i-1,j}\frac{k^1_{i-\frac{1}{2},j}}{h_{1i}\frac{h_{1,i}+h_{1i-1}}{2}} - p_{i,j+1}\frac{k^2_{i,j+\frac{1}{2}}}{h_{2j}\frac{h_{2,j+1}+h_{2j}}{2}} - p_{i,j-1}\frac{k^2_{i,j-\frac{1}{2}}}{h_{2j}\frac{h_{2,j}+h_{2,j-1}}{2}}$$

$$+p_{ij}\left[\frac{k^1_{i+\frac{1}{2},j}}{h_{1i}\frac{h_{1,i+1}+h_{1i}}{2}} + \frac{k^1_{i-\frac{1}{2},j}}{h_{1i}\frac{h_{1i}+h_{1,i-1}}{2}} + \frac{k^2_{i,j+\frac{1}{2}}}{h_{2j}\frac{h_{2,j+1}+h_{2j}}{2}} + \frac{k^2_{i,j-\frac{1}{2}}}{h_{2j}\frac{h_{2j}+h_{2,j-1}}{2}}\right] = \frac{1}{h_{1i}h_{2j}}\int_{T_{ij}} f$$

When $h = h_1 = h_2$ and K are constant, with $k = k^1 = k^2$, this equation results in the standard equation:

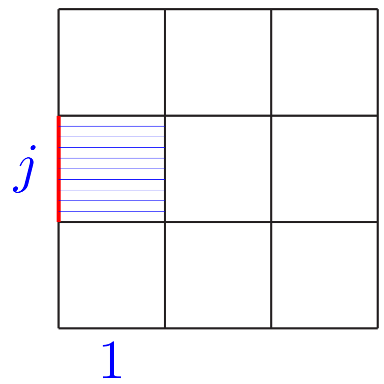
$$\frac{k}{h^2}\left[-p_{i+1,j} - p_{i-1,j} - p_{i,j+1} - p_{i,j-1} + 4p_{i,j}\right] = \frac{1}{h_{1i}h_{2j}}\int_{T_{ij}} f$$

On the Dirichlet boundary Γ_D: $p = \overline{p}$.

Say for instance for the left boundary:

$$u_{\frac{1}{2},j} = -k^1_{\frac{1}{2},j} \frac{p_{1,j} - \overline{p}_j}{\frac{1}{2}h_{1,1}} h_{2j}$$

with $k^1_{\frac{1}{2},j} = k^1_{1,j}$



The corresponding equation is now

$$-p_{2,j} \frac{k^1_{\frac{3}{2},j}}{h_{1,1}\frac{h_{1,2}+h_{1,1}}{2}} - p_{1,j+1} \frac{k^2_{1,j+\frac{1}{2}}}{h_{2,j}\frac{h_{2,j+1}+h_{2,j}}{2}} - p_{1,j-1} \frac{k^2_{1,j-\frac{1}{2}}}{h_{2j}\frac{h_{2,j}+h_{2,j-1}}{2}}$$

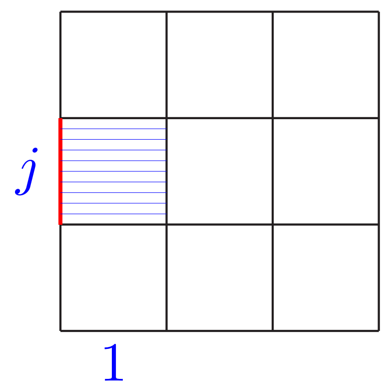
$$+p_{1,1} \left[\frac{k^1_{\frac{3}{2},j}}{h_{1,1}\frac{h_{1,2}+h_{1,1}}{2}} + \frac{k^1_{1,j}}{\frac{h^2_{1,1}}{2}} + \frac{k^2_{1,j+\frac{1}{2}}}{h_{2,j}\frac{h_{2,j+1}+h_{2,j}}{2}} + \frac{k^2_{1,j-\frac{1}{2}}}{h_{2,j}\frac{h_{2,j}+h_{2,j-1}}{2}} \right] =$$

$$\frac{1}{h_{1,1}h_{2,j}} \int_{T_{ij}} f + \overline{p}_j \frac{k^1_{1,j}}{\frac{h^2_{1,1}}{2}}$$

On the Neumann boundary Γ_N: $\vec{u} \cdot \vec{n} = g$.

Say for instance for the left boundary:

$$u_{\frac{1}{2},j} = -g h_{2j}$$



Then we obtain

$$-p_{2,j} \frac{k^1_{\frac{3}{2},j}}{h_{1,1}\frac{h_{1,2}+h_{1,1}}{2}} - p_{1,j+1} \frac{k^2_{1,j+\frac{1}{2}}}{h_{2j}\frac{h_{2,j+1}+h_{2j}}{2}} - p_{1,j-1} \frac{k^2_{1,j-\frac{1}{2}}}{h_{2j}\frac{h_{2,j}+h_{2,j-1}}{2}}$$

$$+p_{1,1}\left[\frac{k^1_{\frac{3}{2},j}}{h_{1,1}\frac{h_{1,2}+h_{1,1}}{2}} + \frac{k^2_{1,j+\frac{1}{2}}}{h_{2j}\frac{h_{2,j+1}+h_{2j}}{2}} + \frac{k^2_{1,j-\frac{1}{2}}}{h_{2j}\frac{h_{2j}+h_{2,j-1}}{2}}\right] =$$

$$\frac{1}{h_{1,1}h_{2j}}\int_{T_{ij}} f - \frac{g}{h_{1,1}}$$

Mixed finite elements vs finite volumes

Assume still that K is diagonal.

Use trapezoidal rule to calculate the coefficients of A. Then

$$\left[\int_{T_{ij}} \frac{1}{k^1_{ij}} \vec{v}_{i+1/2,j} \cdot \vec{v}_{i+1/2,j} + \int_{T_{i+1,j}} \frac{1}{k^1_{i,j+1}} \vec{v}_{i+1/2,j} \cdot \vec{v}_{i+1/2,j}\right] = \frac{1}{2h_{2j}} \left(\frac{h_{1i}}{k^1_{ij}} + \frac{h_{1,i+1}}{k^1_{i+1}} \right)$$

$$\int_{T_{ij}} \frac{1}{k^1_{ij}} \vec{v}_{i-1/2,j} \cdot \vec{v}_{i+1/2,j} = 0$$

$$\int_{T_{i+1,j}} \frac{1}{k^1_{i,j+1}} \vec{v}_{i+3/2,j} \cdot \vec{v}_{i+1/2,j} = 0$$

Therefore the matrix A_V becomes diagonal. Its rows correspond to the equations:

$$u_{i+\frac{1}{2},j} = -\left(\frac{k^1}{h_1}\right)_{i+\frac{1}{2},j} (p_{i+1,j} - p_{i,j})h_{2j}$$

with
$$\left(\frac{k^1}{h_1}\right)_{i+\frac{1}{2},j} = \frac{1}{\frac{1}{2}\left(\frac{h_{1i}}{k^1_{ij}} + \frac{h_{1,i+1}}{k^1_{i,j+1}}\right)} = \text{the harmonic average of } \frac{k^1}{h_1}.$$

This formula for $u_{i+\frac{1}{2},j}$ is slightly different from that given before for a standard finite volume method.

It is natural since one can realize that the coefficient in front of $(p_{i+1,j} - p_{i,j})h_{2j}$ is actually like k^1/h_1 (and not just k_1).

It gives better results in cases where there is also a sharp change in h_1.

What did we actually do to obtain finite volumes from mixed finite elements ?

We approximated a by a_h such that

$$\begin{aligned} a(\vec{u}_h, \vec{v}_h) &= \int_\Omega K^{-1}\vec{u}_h \cdot \vec{v}_h = \sum_{T\in\mathcal{T}_h} \int_T K^{-1}\vec{u}_h \cdot \vec{v}_h \\ a_h(\vec{u}_h, \vec{v}_h) &= \sum_{T\in\mathcal{T}_h} \oint_T K^{-1}\vec{u}_h \cdot \vec{v}_h \end{aligned}$$

where $\oint_T$ is an approximate integral over T calculated with the trapezoidal rule in x_1 for the vertical edges and in x_2 for horizontal edges.

The bilinear form a_h can be rewritten as

$$a_h(\vec{u}_h, \vec{v}_h) = \sum_{T \in \mathcal{T}_h} \sum_{F \subset \partial T} \alpha_{T,F} \int_F \vec{u}_h \cdot \vec{n}_F \int_F \vec{v}_h \cdot \vec{n}_F.$$

with $\alpha_{T(E),E} = \dfrac{1}{2|E|} \dfrac{h^1_{T(E)}}{k^1_{T(E)}}$ for a vertical edge E.

This gives a matrix A_h corresponding to a_h which is diagonal.

$$\begin{aligned}
a_h(\vec{v}_E, \vec{v}_E) &= \alpha_{T_1(E),E} + \alpha_{T_2(E),E}, \quad T_1(E), T_2(E) \in \mathcal{N}(E), \qquad \text{if } E \not\subset \Gamma, \\
a_h(\vec{v}_E, \vec{v}_E) &= \alpha_{T(E),E} \quad \text{if } E \subset \Gamma, \\
a_h(\vec{v}_E, \vec{v}_F) &= 0 \quad \text{if } E \neq F.
\end{aligned}$$

The new approximate formulation is

$$\mathcal{P}^*_{mh} \begin{cases} \text{Find } \vec{u}^*_h \in \mathcal{W}_{hg} \text{ and } p^*_h \in \mathcal{M}_h \text{ such that} \\ a_h(\vec{u}^*_h, \vec{v}_h) - b(\vec{v}_h, p^*_h) = l_{\mathcal{W}}(\vec{v}_h), \qquad \vec{v}_h \in \mathcal{W}_{h0}, \\ b(\vec{u}^*_h, q_h) = l_{\mathcal{M}}(q_h), \qquad q_h \in \mathcal{M}_h. \end{cases}$$

which is equivalent to the cell-centered finite volume formulation on rectangles.

The algebraic system is now

$$\begin{bmatrix} A_h & -{}^tD \\ D & 0 \end{bmatrix} \begin{bmatrix} U^* \\ P^* \end{bmatrix} = \begin{bmatrix} F_v \\ F_q \end{bmatrix}.$$

The row equation associated with $\vec{v}_h = \vec{v}_E$ reads now

$$(\alpha_{T_1(E),E} + \alpha_{T_2(E),E})u_E^* - p_{T_1(E)}^* + p_{T_2(E)}^* = 0, \quad E \not\subset \Gamma,$$
$$\alpha_{T,E}\, u_E^* - p_{T_1(E)}^* + \overline{p}_E = 0, \quad E \subset \Gamma.$$

One can know eliminate U^* to obtain the linear system for P^*

$$(DA_h^{-1}\,{}^tD)\; P* = F_q - DA_h^{-1}F_v$$

where $DA_h^{-1t}D$ is still a sparse matrix (5 diagonals).

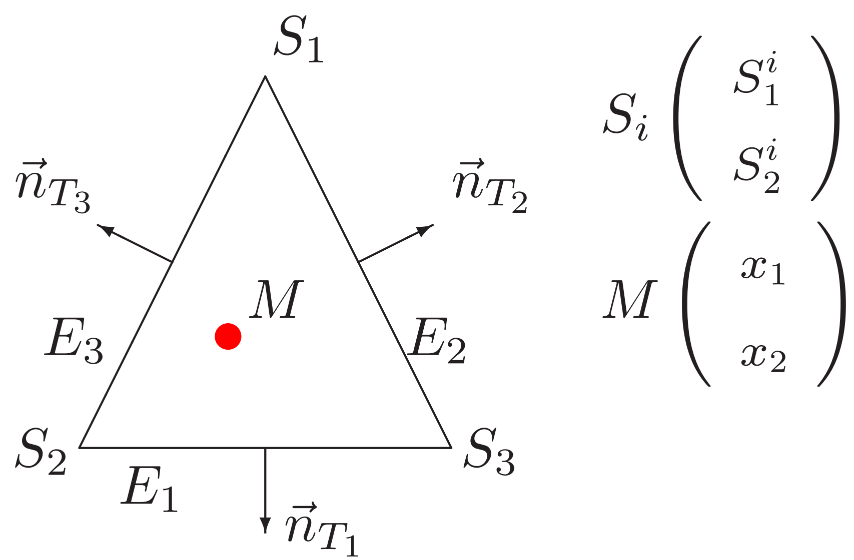
Did we lose accuracy by replacing a by a_h ?

Le cas des triangles

Base de $\vec{\mathcal{W}}_T$:

$$\vec{v}_{T,E_i} = \frac{1}{2|T|}\begin{pmatrix} x_1 - S_1^i \\ x_2 - S_2^i \end{pmatrix} = \frac{1}{2|T|}\vec{S_iM},$$

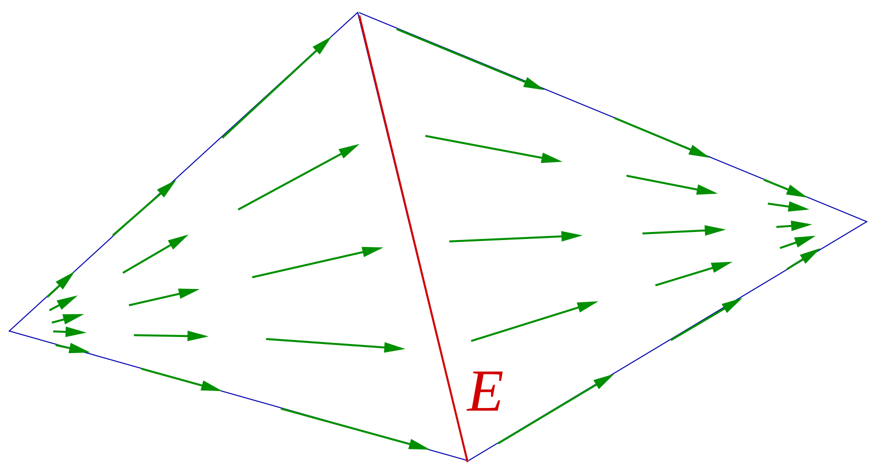
$i = 1, 2, 3$



Les $\vec{v}_{T,E_i}$ vérifient $\displaystyle\int_{E_j} \vec{v}_{T,E_i} \cdot \vec{n}_{T_j} = \delta_{ij}$.

Base de $\mathcal{W}_h$:

$$\vec{v}_E(x) = \begin{cases} & \vec{v}_{T_1(E),E}(x) \quad \text{si } x \in T_1(E) \\ - & \vec{v}_{T_2(E),E}(x) \quad \text{si } x \in T_2(E) \\ & 0 \quad \text{ailleurs} \end{cases}$$



Volumes finis triangulaires

Comme pour les rectangles on approche a par a_h de sorte que

$$
\begin{aligned}
a(\vec{u}_h, \vec{v}_h) &= \sum_{T \in \mathcal{T}_h} \underbrace{\int_T K^{-1} \vec{u}_h \cdot \vec{v}_h}_{a^T(\vec{u}_h, \vec{v}_h)} \\
a_h(\vec{u}_h, \vec{v}_h) &= \sum_{T \in \mathcal{T}_h} \underbrace{\sum_{i=1}^{3} \alpha_{T,E_i} \int_{E_i} \vec{u}_h \cdot \vec{n}_{E_i} \int_{E_i} \vec{v}_h \cdot \vec{n}_{E_i}.}_{a_h^T(\vec{u}_h, \vec{v}_h)}
\end{aligned}
$$

La matrice de a_h est diagonale.

Trouver $\vec{u}_h^* \in \mathcal{W}_h$ et $p_h^* \in M_h$ tels que

$$\begin{array}{lclcl} a_h(\vec{u}_h^*, \vec{v}_h) & -b(p_h^*, \vec{v}_h) & = & g(\vec{v}_h), & \vec{v}_h \in \mathcal{W}_h, \\ b(\vec{u}_h^*, q_h) & & = & f(q_h), & q_h \in \vec{M}_h. \end{array}$$

Le système algébrique s'écrit maintenant

$$\begin{bmatrix} A_h & -{}^tB \\ B & 0 \end{bmatrix} \begin{bmatrix} U^* \\ P^* \end{bmatrix} = \begin{bmatrix} \overline{F} \\ F \end{bmatrix}.$$

La ligne du système correspondant à $\vec{q}_E$ s'écrit maintenant

$$(\alpha_{T_1(E),E} + \alpha_{T_2(E),E}) u_E^* - p_{T_1(E)}^* + p_{T_2(E)}^* = 0, \qquad E \not\subset \partial\Omega,$$

$$\alpha_{T,E}\, u_E^* - p_{T_1(E)}^* + \overline{p}_E = 0, \quad E \subset \partial\Omega.$$

On peut donc maintenant éliminer U^* en maintenant la structure creuse de la matrice en P^*.

Reste à choisir les coefficients α_{T,E_i} de sorte que la précision ne soit pas affectée.

Estimations d'erreur

a et b sont comme avant, vérifiant les hypothèses de continuité, de $\mathcal{V}$-ellipticité, et la condition inf-sup .

Théorème : Hypothèses sur a_h : il existe A^*, α^* indépendantes de h telles que

(H1) $a_h(\vec{u}_h, \vec{v}_h) \leq A^* \|\vec{u}_h\|_{\mathcal{W}} \|\vec{v}_h\|_W, \quad \vec{u}_h, \vec{v}_h \in \mathcal{W}_h$

(H2) $a_h(\vec{v}_h, \vec{v}_h) \geq \alpha^* \|\vec{v}_h\|_W^2, \ \vec{v}_h \in \mathcal{V}_h = \{\vec{v}_h \in \mathcal{W}_h \mid b(\vec{v}_h, q_h) = 0, \ q_h \in \mathcal{M}_h\}.$

Alors il existe C telle que

$$\|\vec{u} - \vec{u}_h^*\|_{\mathcal{W}} + \|p - p_h^*\|_{\mathcal{M}}$$

$$\leq C \left\{ \inf_{\vec{v}_h \in \mathcal{W}_h} \left(\|\vec{u} - \vec{v}_h\|_{\mathcal{W}} + \sup_{\vec{\eta}_h \in \mathcal{W}_h} \frac{a(\vec{v}_h, \vec{\eta}_h) - a_h(\vec{v}_h, \vec{\eta}_h)}{\|\vec{\eta}_h\|_{\mathcal{W}}} \right) + \inf_{q_h \in \mathcal{M}_h} (\|p - q_h\|_{\mathcal{M}}) \right\}.$$

On connait déjà les erreurs d'interpolation :

$$\inf_{q_h \in \mathcal{M}_h} \|p - q_h\|_{\mathcal{M}} \leq Ch\|p\|_{H^1(\Omega)}, \quad \inf_{\vec{v}_h \in \mathcal{W}_h} (\|\vec{u} - \vec{v}_h\|_{\mathcal{W}}) \leq Ch(\|\vec{u}\|_{H_1(\Omega)} + \|\mathrm{div}\vec{u}\|_{H_1(\Omega)}).$$

Il reste à vérifier les hypothèses (H1) et (H2) pour appliquer le théorème, et à évaluer l'erreur $a - a_h$.

Vérification de l'hypothèse (H2)

Les coefficients
$$\alpha_{T,E_1} = -\frac{1}{4|T|}(K_T^{-1}\vec{S_1S_3}) \cdot \vec{S_2S_1},$$
$$\alpha_{T,E_2} = -\frac{1}{4|T|}(K_T^{-1}\vec{S_2S_1}) \cdot \vec{S_3S_2}, \qquad \alpha_{T,E_3} = -\frac{1}{4|T|}(K_T^{-1}\vec{S_3S_2}) \cdot \vec{S_1S_3}.$$

sont choisis de sorte que

Lemme :

$$a^T(\vec{u}_h, \vec{v}_h) = a_h^T(\vec{u}_h, \vec{v}_h), \quad \vec{u}_h, \vec{v}_h \in \mathcal{V}_T = \{\vec{v} \in \mathcal{W}_T; \mathrm{div}\vec{v} = 0\} = (P_0(T))^2. \quad (1)$$

Corollaire : l'hypothèse (H2) est vérifiée.

En effet, pour $\vec{v}_h \in \mathcal{V}_h$,
$$a_h(\vec{v}_h, \vec{v}_h) = \sum_{T \in \mathcal{T}_h} a_h^T(\vec{v}_h, \vec{v}_h) = \sum_{T \in \mathcal{T}_h} a^T(\vec{v}_h, \vec{v}_h) = a(\vec{v}_h, \vec{v}_h) \geq \alpha \|\vec{v}_h\|_{\mathcal{W}}^2.$$

Vérification de l'hypothèse (H1)

On va montrer que

$$|a_h^T(\vec{u}_h, \vec{v}_h)| \leq C\|\vec{u}_h\|_{0,T}\|\vec{v}_h\|_{0,T}, \quad \vec{u}_h, \vec{v}_h \in \mathcal{W}_h \tag{2}$$

ce qui implique (H1).

Passage à l'élément de référence : etc...

Estimation de l'erreur $a - a_h$

Lemme :

$$\inf_{\vec{v}_0 \in (P_0(T))^2} \|\vec{v}_h - \vec{v}_0\|_{0,T}\| \leq Ch\|\text{div}\vec{v}_h\|_{0,T}, \quad \vec{v}_h \in \mathcal{W}_h. \tag{3}$$

En effet, soit $\vec{v}_0 \in \mathcal{V}_T$ tel que $\vec{v}_h = \vec{v}_0 + \beta(\vec{v}_{T,1} + \vec{v}_{T,2} + \vec{v}_{T,3})$ avec $\beta = \dfrac{|T|\text{div}\vec{v}_h}{3}$.

$$\begin{aligned}
\|\vec{v}_h - \vec{v}_0\|_{0,T}^2 &= \beta^2 \frac{1}{4|T|^2} \int_T (S_1\vec{M} + S_2\vec{M} + S_3\vec{M})^2 \\
&= \beta^2 \frac{1}{12|T|} \sum_{i=1,2,3} (S_1\vec{M}_1 + S_2\vec{M}_2 + S_3\vec{M}_3)^2
\end{aligned}$$

où M_1, M_2, M_3 sont les milieux des arêtes opposées aux sommets S_1, S_2, S_3.

etc...

Donc

$$\|\vec{v}_h - \vec{v}_0\|_{0,T}^2 = \frac{1}{144}(\sum_{i=1}^{3} |E_i|^2)\|\mathsf{div}\vec{v}_h\|_{0,T}^2$$

ce qui implique le lemme.

On peut maintenant estimer $a - a_h$.

Proposition :

$$\inf_{\vec{v}_h \in \mathcal{W}_h} \sup_{\vec{\eta}_h \in \mathcal{W}_h} \frac{a(\vec{v}_h, \vec{\eta}_h) - a_h(\vec{v}_h, \vec{\eta}_h)}{\|\vec{\eta}_h\|_{\mathcal{W}}} \leq Ch\|\vec{u}\|_{\mathcal{W}}. \tag{4}$$

Preuve : Pour tout $\vec{v}_h, \vec{\eta}_h \in \mathcal{W}_T$, il existe $\vec{v}_0, \vec{\eta}_0 \in \mathcal{V}_T$ vérifiant

$$\|\vec{v}_h - \vec{v}_0\|_{0,T} \leq Ch\|\text{div}\vec{v}_h\|_{0,T}, \quad \|\vec{\eta}_h - \vec{\eta}_0\|_{0,T} \leq Ch\|\text{div}\vec{\eta}_h\|_{0,T}. \tag{5}$$

Alors

$$\begin{aligned} a^T(\vec{v}_h, \vec{\eta}_h) - a_h^T(\vec{v}_h, \vec{\eta}_h) = \quad & a^T(\vec{v}_h, \vec{\eta}_h - \vec{\eta}_0) + a^T(\vec{v}_h - \vec{v}_0, \vec{\eta}_0) + a^T(\vec{v}_0, \vec{\eta}_0) \\ & -[a_h^T(\vec{v}_h, \vec{\eta}_h - \vec{\eta}_0) + a_h^T(\vec{v}_h - \vec{v}_0, \vec{\eta}_0) + a_h^T(\vec{v}_0, \vec{\eta}_0)] \end{aligned}$$

L'égalité (1) et les inégalités (2),(5) impliquent

$$\begin{aligned} |a^T(\vec{v}_h,\vec{\eta}_h) - a_h^T(\vec{v}_h,\vec{\eta}_h)| &\leq C(\|\vec{v}_h\|_{0,T}\|\vec{\eta}_h - \vec{\eta}_0\|_{0,T} + \|\vec{v}_h - \vec{v}_0\|_{0,T}\|\vec{\eta}_0\|_{0,T}) \\ &\leq Ch_T(\|\vec{v}_h\|_{\mathcal{W}_T}\|\vec{\eta}_h\|_{\mathcal{W}_T}). \end{aligned}$$

Donc, pour tout $\vec{v}_h, \vec{\eta}_h \in \mathcal{W}_h$ on a

$$\begin{aligned} a(\vec{v}_h,\vec{\eta}_h) - a_h(\vec{v}_h,\vec{\eta}_h)| &\leq Ch(\|\vec{v}_h\|_{\mathcal{W}}\|\vec{\eta}_h\|_{\mathcal{W}}), \\ \sup_{\vec{\eta}_h\in\mathcal{W}_h} \frac{a(\vec{v}_h,\vec{\eta}_h) - a_h(\vec{v}_h,\vec{\eta}_h)}{\|\vec{\eta}_h\|_{\mathcal{W}}} &\leq Ch\|\vec{v}_h\|_{\mathcal{W}}. \end{aligned}$$

Si $\vec{v}_h = \Pi_h\vec{u}$ alors

$$\sup_{\vec{\eta}_h\in\mathcal{W}_h} \frac{a(\Pi_h\vec{u},\vec{\eta}_h) - a_h(\Pi_h\vec{u},\vec{\eta}_h)}{\|\vec{\eta}_h\|_{\mathcal{W}}} \leq Ch\|\Pi_h\vec{u}\|_{\mathcal{W}} \leq Ch\|\vec{u}\|_{\mathcal{W}}$$

et l'inégalité (4) est vraie.

Fin des estimations d'erreur

Grâce aux estimations d'interpolation et à l'estimation (4), le théorème 1 nous permet de conclure :

$$\|\vec{u} - \vec{u}_h^*\|_{\mathcal{W}} + \|p - p_h^*\|_{\mathcal{M}} \leq Ch(\|p\|_{1,\Omega} + \|\vec{u}\|_{1,\Omega} + \|\mathrm{div}\vec{u}\|_{1,\Omega}).$$

Remarque : Pour que l'analyse ci-dessous fonctionne il faut que les coefficients $\alpha_{T,E_i}, i = 1, 2, 3$ soient strictement positifs

$\Longrightarrow$ les angles des triangles de $\mathcal{T}_h$ doivent être tous aigus.

Qualities and difficulties of the finite volume method

- On rectangles it is simple, leads to a positive-definite matrix, and satisfies the maximum principle

- Difficult to formulate for differmed rectangles and triangles

- For triangles restriction on the meshes

- Complicated formulation when K is not diagonal

OTHER CELL-CENTERED DISCRETIZATION METHODS

Mixed-hybrid finite elements

Instead of calculating $\vec{u}_h \in \mathcal{W}_h$, we now calculate

$$\vec{u}_h^\star \in \mathcal{W}_h^\star = \{\vec{v}_h \in (L^2(\Omega))^2; \vec{v}_h|T \in \mathcal{W}_T, T \in \mathcal{T}_h\}$$

Functions of $\mathcal{W}_h^\star$ are not required to have their flux continuous across the edges.

Continuity of the flux will now be written explicitly.

We need also

$$\mathcal{N}_h = \{\mu_h \in \Pi_{E\in\mathcal{A}_h}\, \mu_E,\ \mu_E \in \mathbb{R}\}.$$

The mixed-hybrid formulation is

Find $\vec{u}_h^\star \in \mathcal{W}_{hg}^\star, p_h^\star \in \mathcal{M}_h, \lambda_h \in \mathcal{N}_h$ such that

$$\int_T K^{-1}\vec{u}_h^\star \cdot \vec{v}_h - \int_T p_h^\star \,\mathrm{div}\, \vec{v}_h + \sum_{E\in\partial T} \int_E \lambda_h \vec{v}_h \cdot \vec{n}_T = \int_{\Gamma_D} -\overline{p}\, \vec{v}_h \cdot \vec{n}_T,$$

$$\vec{v}_h \in \mathcal{W}_h^\star, T \in \mathcal{T}_h$$

$$\int_T \mathrm{div}\, \vec{u}_h \; q_h = \int_T f \; q_h, \quad q_h \in \mathcal{M}_h, \; T \in \mathcal{T}_h$$

$$-\sum_{T\in\mathcal{T}_h, \partial T \supset E} \vec{u}_h^\star \cdot \vec{n}_T \; \mu_h = 0, \quad E \in \mathcal{A}_h, E \not\subset \Gamma, \mu_h \in \mathcal{N}_h$$

$$\vec{u}_h^\star \cdot \vec{n}|_E = g|E|, \quad E \subset \Gamma_N,$$

$$\lambda_h|_E = \overline{p}, \quad E \subset \Gamma_D.$$

λ_h represents a trace of the pressure on the edges $E \in \mathcal{A}_h$.

We check easily that $p_h^\star = p_h, \vec{u}_h^\star|_T = \vec{u}_h|_T, \; T \in \mathcal{T}_h$.

The linear system

$$\begin{bmatrix} A^\star & -{}^tD & -{}^tB \\ D & 0 & 0 \\ B & 0 & I_D \end{bmatrix} \begin{bmatrix} U^\star \\ P \\ \Lambda \end{bmatrix} = \begin{bmatrix} F_v \\ F_q \\ F_\mu \end{bmatrix}$$

$A^\star$ is block diagonal; we can eliminate $U^\star = A^{\star(-1)}(F_v + {}^tDP + {}^tB\Lambda)$ to get

$$\begin{bmatrix} DA^{\star(-1)}\,{}^tD & DA^{\star(-1)}\,{}^tB \\ BA^{\star(-1)}\,{}^tD & BA^{\star(-1)}\,{}^tB + I_D \end{bmatrix} \begin{bmatrix} P \\ \Lambda \end{bmatrix} = \begin{bmatrix} F_q - DA^{\star(-1)}F_v \\ F_\mu - BA^{\star(-1)}F_v \end{bmatrix}$$

The matrix $DA^{\star(-1)}\,{}^tD$ is diagonal, so we can eliminate P:
$P = (DA^{\star(-1)}\,{}^tD)^{-1}[F_q - DA^{\star(-1)}F_v - (DA^{\star(-1)}\,{}^tB)\Lambda]$ to obtain

$$H\Lambda = G \quad (H \text{ sparse})$$

where $\quad H = BA^{\star(-1)}\,{}^{t}B + I_D - (BA^{\star(-1)}\,{}^{t}D)(DA^{\star(-1)}\,{}^{t}D)^{-1}(DA^{\star(-1)}\,{}^{t}B)$

$$G = F_\mu - BA^{\star(-1)} - (BA^{\star(-1)}\,{}^{t}D)(DA^{\star(-1)}\,{}^{t}D)^{-1}(F_q - DA^{\star(-1)}F_v)$$

Properties of the matrix H

- H is sparse

The number of nonzeros in the line E is equal to the number of neighbouring edges + 1 (for E itself) (7 for a rectangular mesh).

- H is positive definite

To prove it, assuming $(H\Lambda, \Lambda) = 0$ we have to show that this implies $\Lambda = 0$. Then

$$((BA^{\star(-1)}\,{}^tB + I_D)\Lambda, \Lambda) - ((BA^{\star(-1)}\,{}^tD)(DA^{\star(-1)}\,{}^tD)^{-1}(DA^{\star(-1)}\,{}^tB)\Lambda, \Lambda) = 0$$

Introduce $P = (DA^{\star(-1)}\,{}^tD)^{-1}(-(DA^{\star(-1)}\,{}^tB)\Lambda)$. We obtain

$$(A^{\star(-1)}\,{}^tB\Lambda, {}^tB\Lambda) + (I_D\Lambda, \Lambda) - (A^{\star(-1)}\,{}^tDP, B\Lambda) = 0$$

But equation for P implies that

$$((DA^{\star(-1)}\,{}^tD)P, P) + ((DA^{\star(-1)}\,{}^tB)\Lambda, P) = 0.$$

Adding to the previous equation gives

$$(A^{\star(-1)}({}^tDP + {}^tB\Lambda), {}^tDP + {}^tB\Lambda) + (I_D\Lambda, \Lambda) = 0$$

Since $A^{\star(-1)}$ is positive definite and I_D is positive semi-definite, this implies that ${}^tDP + {}^tB\Lambda = 0$ and $\lambda_E = 0, E \subset \Gamma_D$.

Equation ${}^tDP + {}^tB\Lambda = 0$ says actually that

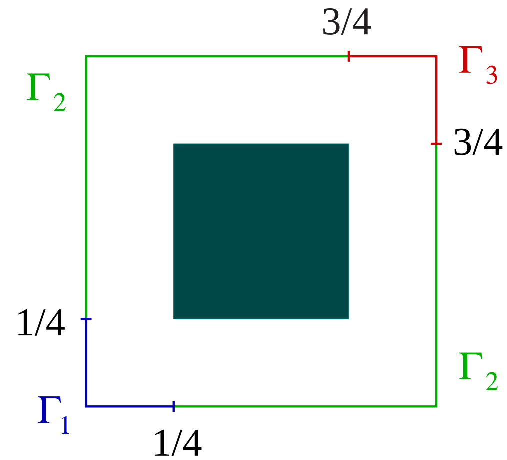
$$P_T - \lambda_E = 0, E \supset \partial T, T \in \mathcal{T}_h$$

which means that the pressure is constant over Ω.

But from $\lambda_E = 0, E \subset \Gamma_D$ it follows that $P = 0$ and $\Lambda = 0$.

A PROJECT

The problem



div $\vec{u} = f, \quad \vec{u} = -K\ \overrightarrow{\text{grad}}\ p,$ in $\Omega = (0.1) \times (0,1)$

$p = 100$ on Γ_1, $\quad \vec{u} \cdot \vec{n} = 0$ on Γ_2, $\quad \vec{u} \cdot \vec{n} = 10$ on Γ_3.

1. Solve for $K = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}, \quad K = \begin{bmatrix} 10 & 0 \\ 0 & 100 \end{bmatrix}, \quad K = \begin{bmatrix} 10 & 9 \\ 9 & 10 \end{bmatrix}$

2. Solve for $K = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ in the shaded area, $K = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$ outside the shaded area.

3. Calculate the errors $\|p - p_h\|_0$, $\|u - u_h\|_0$.

A remark on checking the program

One check of the program follows from choosing $u \in \mathcal{W}_h$.

Since $\mathcal{W}_h \subset \mathcal{W}, \mathcal{M}_h \subset \mathcal{M}$ problem $\mathcal{P}_m$ implies

$$\begin{cases} a(\vec{u}, \vec{v}_h) & -b(\vec{v}_h, p) & = & l_{\mathcal{W}}(\vec{v}_h), & \vec{v} \in \mathcal{W}_{h0}, \\ b(\vec{u}, q_h) & & = & l_{\mathcal{M}}(q_h), & q_h \in \mathcal{M}_h. \end{cases}$$

Denoting $\Pi_h p$ the L^2-projection of p, since div is a mapping from $\mathcal{W}_h$ onto $\mathcal{M}_h$, we have $\int_\Omega \mathrm{div}\vec{v}_h(p - \Pi_h p) = 0$ for all $\vec{v}_h \in \mathcal{W}_h$.

Therefore $\vec{u} \in \mathcal{W}_h, \Pi_h p \in \mathcal{M}_h$ is the unique solution to the equations :

$$\begin{cases} a(\vec{u}, \vec{v}_h) & -b(\vec{v}_h, \Pi_h p) & = & l_{\mathcal{W}}(\vec{v}_h), & \vec{v} \in \mathcal{W}_{h0}, \\ b(\vec{u}, q_h) & & = & l_{\mathcal{M}}(q_h), & q_h \in \mathcal{M}_h. \end{cases}$$

Thus, to check the program,

1. choose $\vec{u} \in \mathcal{W}_h$
2. calculate directly $\displaystyle\int_E \vec{u} \cdot \vec{n}_E,\ E \in \mathcal{A}_h$ as well as $\displaystyle\frac{1}{|T|}\int_T p,\ T \in \mathcal{T}_h$.

These numbers should be equal to $u_{E,E\in\mathcal{A}_h},\ p_{T,T\in\mathcal{T}_h}$ calculated by the program up to the machine accuracy.