Abstract. This paper shows equilibrium existence for an infinite–horizon exchange economy with incomplete markets of real assets when default is allowed. Borrowers are required to constitute collateral in terms of durable goods and face credit constraints that depend on their past default. When credit constraints are inelastic, Ponzi schemes are ruled out and an equilibrium exists.

JEL Classification: D52, D91.

Keywords: Equilibrium, Incomplete markets, Default, Collateral, Credit constraints, Ponzi schemes.

I am indebted to M. Magill, M.R. Pascoa and M. Quinzii for valuable comments and stimulating conversations at the SAET 2009 conference in Ischia/Italy. I thank J.P. Torres-Martínez for useful comments on a previous version of this paper. I would also like to thank L.H. Braido for access to his work prior to its publication and for interesting conversations on this problem.
1. Introduction.

When default is allowed, some mechanisms must be imposed that would urge borrowers to pay back at least a part of their debts, otherwise, agents would have no incentive to pay. Various approaches have been introduced in the literature in attempts to urge borrowers to fulfill their promises either partially or fully. A first mechanism, which guarantees at least a partial reimbursement of the debt, requires borrowers to constitute collateral in terms of durable goods. Lack of payment results in the seizure of the collateral by the lenders. This approach was introduced by Geanakoplos and Zame (1995) for a finite–horizon economy and used among others by Araujo et al. (2002), Kubler and Schmedders (2003) and Ferreira and Torres-Martinez (2010) for infinite–horizon economies. A second mechanism assumes that defaulters face credit constraints that depend on their past default. Using such a mechanism, Sabarwal (2003) proves equilibrium existence for a finite–horizon economy with incomplete markets and a continuum of agents. Braido (2008) shows the existence of a Markovian equilibrium for an infinite–horizon with a static stochastic structure and numeraire assets when defaulters face credit constraints. This paper studies an infinite–horizon exchange economy with incomplete markets of real assets when borrowers are required to constitute collateral in terms of durable goods and defaulters face credit constraints that depend on their default history.

The main problem that arises in infinite–horizon economies is the possibility of the so-called Ponzi schemes (e.g., Levine (1989) for more details). Even if the system of prices does not offer arbitrage, the decision problem of the agents may not have a solution. More precisely, for any no-arbitrage system of prices, an infinitely-lived agent can renew his credit and postpone the repayment of his debt until infinity. Thus, to have a solution for the maximization problems of agents with monotone preferences, one must impose some mechanism that avoids such ruses.

In the literature of incomplete markets without default, there are at least two manners to avoid such Ponzi schemes. The first being, the debt constraints introduced by Levine (1989) and developed by Levine and Zame (1996), which imposes that at each node, the value of what each agent can borrow is bounded by a system of debt constraints. The second one, called the transversality condition, limits the rate of growth of debts (e.g., Kehoe (1989) and Magill and Quinzii (1994)). In the deterministic case, or in other words when each node of the event-tree has a unique immediate successor, this second condition imposes that debts grow asymptotically slower than the rate of interest.

In an infinite horizon real incomplete market model with default, Araujo et al. (2002) prove that collateral avoid Ponzi schemes when collateral repossession is the unique default enforcement mechanism. The authors do not impose either a transversality condition or debt constraints to limit the possible debt amount. This is because it appears that the obligation of constituting collateral in terms of durable goods whenever an asset is sold will limit the asymptotic explosion of the
debt. Firstly, short-sales become bounded, node by node. Secondly, since there are no additional default penalties, each seller delivers exactly the minimum between his debts and the value of the depreciated collateral. Therefore, the no-arbitrage condition requires borrowed values to be less than the value of the constituted collateral and therefore, bounded from above by a uniform upper bound on endowments of durable goods. In Araujo et al. (2002), collateral repossession is the unique default enforcement mechanism. Such a treatment of default is not fully convincing since default does not affect a household’s ability to borrow in the future and so does not lead to any direct reduction in consumption at the time of default.

Pascoa and Seghir (2009) prove that Ponzi schemes may reappear if an additional default punishment is introduced besides collateral repossession as harsh default penalties may induce effective payments over collateral recollection values. In this event, loans may be larger than collateral costs and agents may prefer doing a Ponzi scheme rather than defaulting and giving up the collateral. The additional default punishment used by Pascoa and Seghir (2009) consists of linear and time–node additively separable utility penalties proportional to the amount of default. These utility penalties were introduced by Shubik and Wilson (1977) and used among others by Dubey et al. (1990), Zame (1993), Araujo et al. (1996), Araujo et al. (1998) and Dubey et al. (2005). Dubey et al. (2005) interpret these utility penalties as “the sum of third party punishment, pangs of conscience, (unmodeled) reputation losses, and (unmodeled) garnishing of future income”. As pointed out by Sabarwal (2003), with such utility penalties, which exist only in the consumer’s psyche, the lender has no legal recourse for debt recovery. Utility penalties also rule out the effect of an agent’s default on his future access to credit markets. In addition, when utility penalties are imposed, present default leads to a direct reduction in utility rather than a reduction in future consumption.

In this paper, I address, in a general equilibrium framework, the actual credit market practices where present default affects a household’s ability to borrow in the future, leading to a direct reduction in consumption at the time of default. More precisely, despite the seizure of the constituted collateral in case of default, defaulters face credit constraints that depend on the amount of their past default. Sabarwal (2003) uses such credit limits in a finite–horizon model with a continuum of agents. However, this paper is independent and there are many valuable differences between the two papers. First, Sabarwal (2003) introduces credit limits mainly to guarantee that short–sales are bounded, node by node, and this is sufficient to assure equilibrium existence in a finite–horizon economy. In our infinite–horizon economy, the obligation of constituting collateral in terms of durable goods of limited endowments endogenously guarantees that short–sales are bounded, node by node, but this is not sufficient to guarantee equilibrium existence in an infinite–horizon economy as agents may end up doing Ponzi schemes. Second, in order to protect the lenders against total default, Sabarwal (2003) assumes that a fraction of debtors’ income can be confiscated and given
to the lenders in case of default. In our model, lenders are partially protected from such a total default as they receive at least the collateral in case of default. Moreover, in order to protect the debtors, Sabrawal (2003) introduces a bankruptcy law ensuring that a fraction of a debtor’s income (i.e.: exemption) cannot be seized. The possibility of confiscating a part of defaulters’ income together with the introduction of a bankruptcy law led to non-convexity of the model in Sabarwal (2003). In order to guarantee equilibrium existence, the author considers a continuum of agents with an atomless distribution. For an infinite–horizon model with a static stochastic structure, Braido (2008) proves the existence of an ergodic Markovian equilibrium when borrowers are not required to constitute collateral and face credit constraints that depend on their past default. To this end, the author assumes that the credit constraints penalizing default are uniformly bounded along the event-tree. This assumption implies that short–sales are uniformly bounded and, therefore, Ponzi schemes are (exogenously) ruled out.

As emphasized by Pascoa and Seghir (2009) and Ferreira and Torres-Martinez (2010), Ponzi schemes may reappear when an additional enforcement mechanism is added to the collateral repossession. However, I prove that the existence of equilibrium and the occurrence of Ponzi schemes depend on how fast credit constraints change with respect to default. More precisely, without imposing any uniform upper bound on the short–sales, I show that (i) agents may end up doing Ponzi schemes if the credit constraints penalizing default are elastic (i.e.: when default on some asset increases, credit opportunities decrease at a higher rate) and (ii) an equilibrium exists when these constraints are inelastic (i.e.: when default on some asset increases, credit opportunities decrease at a lower rate).

The paper is organized as follows. The model is presented in Section 2 and the concepts of equilibrium and non–trivial equilibrium are defined in Section 3. Section 4 covers the possibility of doing Ponzi schemes when collateral requirement and credit constraints penalizing default coexist. In Section 5, an assumption is introduced on how inelastic penalties should be for an equilibrium to exist. Section 5 also provides an example illustrating how agents may end up doing Ponzi schemes in the presence of elastic credit constraints. Finally, an Appendix is devoted to proofs.

2. The Model.

Stochastic Structure.

We consider a discrete time economy with infinite horizon and uncertainty. The stochastic structure of this model is described by an infinite tree with a unique root and finitely many branches at each node. Formally, let $\mathcal{T} = \{0, 1, \ldots\}$ be the set of dates and let $\mathcal{F}_t$ be the finite set of histories that may occur up to time $t$. 
A pair $\xi = (t, \sigma)$ where $t \in T$ and $\sigma \in F_t$ is called node and $t(\xi) = t$ is the date of node $\xi$. The set $D$ consisting of all nodes is called the event-tree.

A node $\xi' = (t', \sigma')$ is said to succeed (resp. strictly) node $\xi = (t, \sigma)$ if $t' \geq t$ (resp. $t' > t$) and $\sigma' \subset \sigma$. We write $\xi' \preceq \xi$ (resp. $\xi' > \xi$). Let $\xi \in D$. We will denote by:

- $D(\xi)$ the subtree of the nodes which succeed $\xi$,
- $D^+(\xi) = \{ \xi' \in D | \xi' > \xi \}$ the set of the strict successors of $\xi$,
- $D_T(\xi)$ the subset of nodes of $D(\xi)$ at date $T$,
- $D^T(\xi)$ the subset of nodes of $D(\xi)$ between $t(\xi)$ and $T$,
- $\xi^+ = \{ \eta \in D(\xi) | t(\eta) = t(\xi) + 1 \}$ the set of immediate successors of $\xi$. The number of elements of $\xi^+$, called the branching number, is assumed to be finite.

If $\xi = (t, \sigma)$, $t \geq 1$, the unique node $\xi^- = (t - 1, \sigma')$, $\sigma \subset \sigma'$ is called the predecessor of $\xi$.

When $\xi$ is the initial node, denoted $\xi_0$, the notations are simplified to $D^+$, $D_T$, $D^T$.

**Commodity, Financial and Demographic Structures.**

At each node $\xi \in D$, a finite number $G$ of physical goods (possibly durable), indexed by $g = 1, \ldots, G$, are traded on spot markets. The structure of depreciation in the event-tree is given by a collection of $G \times G$-matrices $Y := \{ Y(\xi) \}_{\xi \in D}$. As in Araujo et al. (2002), we assume that $Y(\xi)$ is a diagonal matrix, $(\text{diag}(a(\xi, g)))$, for each node $\xi \in D$. A commodity $g \in G$ is durable at node $\xi \in D$ if $a(\xi, g)$ is different from zero and perishes at $\xi$ otherwise.

At each node of the event-tree, there is a set $J(\xi)$ consisting of a finite number $i(\xi)$ of one-period real assets, available for intertemporal transaction and insurance. Let $A^i(\xi) \in \mathbb{R}_+^{G \times i(\xi)}$ be the return, at node $\xi$, in quantities of the $G$ goods, of one unit of the asset $j \in J(\xi^-)$. We denote $A(\xi) = (A^i(\xi))_{i \in J(\xi^-)}$ and $A := \prod_{\xi \in D} A(\xi)$.

At each node $\xi \in D$, a commodity $g \in G$ and an asset $j \in J(\xi)$ are transacted at prices $p(\xi, g)$ and $q_i(\xi)$ respectively. Denote by $p(\xi) = (p(\xi, g), g \in G)$ and $q(\xi) = (q(\xi, j), j \in J(\xi)) \in \mathbb{R}_+^{i(\xi)}$.

The demographic structure of the model is given by a finite set $I$ of infinitely-lived agents. The cardinality of $I$ will be denoted by $|I|$. At each node $\xi$, an agent $i \in I$ chooses:

(i) a consumption $x^i(\xi)$ in $X^i(\xi)$. We denote by $X^i = \prod_{\xi \in D} X^i(\xi)$,

(ii) a portfolio $z^i(\xi) := (z^i_j(\xi), j \in J(\xi))$, with $z^i(\xi) = \theta^i(\xi) - \varphi^i(\xi)$ where:

- $\theta^i(\xi) := (\theta^i_j(\xi), j \in J(\xi)) \in \mathbb{R}_+^{i(\xi)}$ are the quantities of assets bought by $i$ at node $\xi$,
- $\varphi^i(\xi) := (\varphi^i_j(\xi), j \in J(\xi)) \in \mathbb{R}_+^{i(\xi)}$ is the short-sale of assets by $i$ at node $\xi$.

At each node $\xi \in D$, aside his choices of consumption and portfolio, agent $i \in I$ chooses his default $\Delta^i(\xi) = (\Delta^i_j(\xi), j \in J(\xi^-)) \in \mathbb{R}_+^{i(\xi^-)}$. The preferences of an agent $i \in I$ are represented by the utility function $U^i : X^i \rightarrow \mathbb{R}_+$ defined for each $x^i \in X^i$ by: $U^i(x^i) = \sum_{\xi \in D} v^i_\xi(x^i(\xi))$. 
Collateral requirement and credit constraints.

As in Geanakoplos and Zame (1995) and Araujo et al. (2002), each seller of one unit of an asset $j \in J(\xi)$ is required to constitute a collateral $C_j^\xi := (C^j_g, g \in G) \in \mathbb{R}^G_+ \setminus \{0\}$, exogenously given. This collateral is seized and given to the lenders in case of default.

Beside the seizure of his collateral, a borrower $i \in I$, whose default at a node $\xi \in D$ is $\Delta^i(\xi)$ (induced by his asset sales at $\xi^-$), is penalized by facing, on each asset $j \in J(\xi)$, narrow credit constraint functions as follows:

\[
\varphi^i_j(\xi) \leq F^{i,j}_\xi \left( \Delta^i(\xi) \right),
\]

where for each node $\xi \in D$, for each asset $j \in J(\xi)$, $F^{i,j}_\xi : \mathbb{R}^{i(\xi^-)}_+ \rightarrow \mathbb{R}$ is a function that defines the credit constraints penalizing agent $i$’s default. Note that, the credit constraint functions are exogenously given for agents. However, borrowers endogenously choose which credit limit to face by choosing their default level. One can interpret these credit constraint functions as credit limits exogenously chosen by a central institution such as a benevolent central planner for instance.

**Remark 1.** When for each node $\xi \in D$, for each asset $j \in J(\xi)$ and for each agent $i \in I$, the function $F^{i,j}_\xi$ is constant (i.e.: credit opportunities do not change as default changes) with nonnegative values, then each agent $i$ will deliver the minimum between his debt and the value of the depreciated collateral. That is,

\[
\Delta^i_j(\xi) = p(\xi)A^j(\xi)\varphi^i_j(\xi^-) - \min \{p(\xi)A^j(\xi), p(\xi)Y(\xi)C^j(\xi^-)\} \varphi^i_j(\xi^-).
\]

Thus, in such a case, agents’ behavior will be similar to that presented by Araujo et al. (2002). However, the set of equilibrium allocations does not necessarily coincide in both models. Despite this, we obtain the same set of equilibrium allocations as in Araujo et al. (2002) when the function $F^{i,j}_\xi$ is a high enough positive constant. Since the credit constraint functions $F^{i,j}_\xi$ are not necessarily constant in our model, borrowers may choose to deliver more than the minimum between the original promise and the value of the depreciated collateral in order to face wider credit constraints. Thus, in our model, and unlike Araujo et al. (2002), at a node $\xi \in D$, the default $\Delta^i_j(\xi)$ of an agent $i$ on an asset $j \in J(\xi^-)$ satisfies:

\[
0 \leq \Delta^i_j(\xi) \leq p(\xi)A^j(\xi)\varphi^i_j(\xi^-) - \min \{p(\xi)A^j(\xi), p(\xi)Y(\xi)C^j(\xi^-)\} \varphi^i_j(\xi^-).
\]

In view of the anonymity of the markets, lenders do not know what the payment of each individual borrower will be. As in Dubey et al. (2005), we introduce variables representing the expected deliveries of the sellers. Formally, let $(K^j(\xi) \in [0,1], \xi \in D, j \in J(\xi^-))$ be the expected delivery rate on asset $j$ at node $\xi$. This variable will be determined endogenously in equilibrium by the market forces of supply and demand.
We define the Economy $\mathcal{E}$ as follows: $\mathcal{E} := \{(ω^i, F^i, U^i)_{i \in I}, A_i, (C^i(ξ))_{ξ \in D}, Y\).

3. Definitions and Assumptions.

3.1. Definitions: Budget sets and Equilibrium.

Definition 1. [Budget sets]

Given $(p, q, K)$, the budget set $B^i(p, q, K)$ of an agent $i \in I$ is the set of $(x^i, θ^i, φ^i, Δ^i)$ in $\mathbb{R}_+^{G×D} \times \prod_{ξ \in D} \mathbb{R}_{+}^{i(ξ)} \times \prod_{ξ \in D} \mathbb{R}_{+}^{i(ξ)×G}$ verifying:

\[
(2) \quad p(ξ, θ^i(ξ)) \cdot (x^i(ξ) − φ^i(ξ)) + p(ξ, θ^i(ξ)) \cdot φ^i(ξ) + q(ξ) \cdot (θ^i(ξ) − φ^i(ξ)) \leq 0,
\]

and ∀ξ ∈ $D \setminus \{ξ_0\}$,

\[
(3) \quad p(ξ) \cdot (x^i(ξ) − φ^i(ξ)) + p(ξ, C(ξ))φ^i(ξ) + q(ξ) \cdot (θ^i(ξ) − φ^i(ξ)) \leq 0.
\]

\[
(4) \quad φ^i_j(ξ) \leq F^i_j(Δ^i(ξ)), \quad ∀j \in J(ξ).
\]

Let us adopt the following normalization: for each node $ξ \in D$, $\|p(ξ)\|_1 + \|q(ξ)\|_1 = 1$. We denote by $Δ^{n−1} = \{x ∈ \mathbb{R}_+^n : \|x\|_1 = 1\}$.

Definition 2. [Equilibrium]

An equilibrium of $\mathcal{E}$ is a vector $(\overline{p}, \overline{q}, \overline{K}, (\overline{x^i}, \overline{θ^i}, \overline{φ^i}, \overline{Δ^i})_{i \in I})$ such that $\overline{p}(ξ) > 0$ at any node $ξ \in D$ and verifying:

(i) For each agent $i \in I$, $(\overline{x^i}, \overline{θ^i}, \overline{φ^i}, \overline{Δ^i}) \in \text{Argmax} U^i(x)$ over $B^i(\overline{p}, \overline{q}, \overline{K})$,

(ii) $\sum_{i ∈ I} θ^i(ξ_0) = \sum_{i ∈ I} φ^i(ξ_0)$,

(iii) $\sum_{i ∈ I} [x^i(ξ) + C(ξ)φ^i(ξ)] = \sum_{i ∈ I} [ω^i(ξ) + Y(ξ)x^i(ξ) + Y(ξ)C(ξ)φ^i(ξ)], \quad ∀ξ \in D \setminus \{0\},

(iv) $\sum_{i ∈ I} \overline{θ}^i = \sum_{i ∈ I} \overline{φ}^i$,

(v) $∀ξ \in D \setminus \{ξ_0\}, \quad ∀j \in J(ξ), \sum_{i ∈ I} Δ^i_j(ξ) = p(ξ)A^i_j(ξ) \sum_{i ∈ I} [φ^i_j(ξ) − K^i_j(ξ)\overline{θ}^i_j(ξ)]$.

Conditions (i)–(iv) are classical conditions. Condition (v) says that, at each node and for each asset, the total default made by the borrowers is equal to the total debt minus the total deliveries expected by the lenders.

Remark 2. It is possible to prove the existence of an (pure spot market) equilibrium in a trivial way when returns from asset purchases are endogenous (see Dubey et al. (2005), Pascoa and Seghir (2009), Steinert and Torres-Martinez (2007), to name a few). Indeed, if for any $ξ ∈ D$ and $j ∈ J(ξ)$,
asset prices and expected delivery rates \( (q(\xi, j), \bar{K}^i(\mu))_{\mu \in \xi^+} \) are zero, then any spot market equilibrium constitutes an equilibrium for the economy. That is, if agents are over-pessimistic (i.e.: borrowers are expected to make zero payments), there will be no financial transaction (see Steinert and Torres-Martinez (2007) for more details). To overcome the problem of the absence of financial trade as a consequence of zero delivery rates, the existence of an equilibrium in which expected delivery rates are strictly positive needs to be guaranteed. That is an equilibrium in which either there is financial trade or delivery rates are nonnull needs to be secured. On the other hand, if the credit constraint functions have nonpositive values, then there will be no financial trade in equilibrium. This brings about the following definition.

**Definition 3.** [Non-trivial equilibrium] 
A non–trivial equilibrium \( (\vec{\eta}, \vec{\mu}, \vec{\chi}, (\bar{\mu}, \bar{\sigma}, \bar{\nu})_{\mu \in \xi^+}) \) of \( E \) is an equilibrium such that for any \( (\xi, j) \), we have \( (\bar{\eta}_j(\xi), \bar{\sigma}_j(\xi)) \) different from zero or \( \bar{R}^j(\xi) > 0 \).

3.2. Assumptions. We make on \( E \) the following assumptions:

**Assumption [A1].** \( \forall i \in I, \forall \xi \in D, \) the function \( v_i^G : \mathbb{R}_{++}^G \rightarrow \mathbb{R} \) is continuous, monotone\(^1\) and concave with \( v_i^G(0) = 0 \). Moreover, \( \forall i \in I, \forall \alpha \in \mathbb{R}_{++}^G, \) \( \sum_{\xi \in D} v_i^G(\alpha) \) is finite.

**Assumption [A2].** For each agent \( i \in I \), \( \omega^i \in \mathbb{R}_{++}^{G \times D} \) and there exists \( W \in \mathbb{R}_{++} \) such that \( \forall i \in I, \forall \xi \in D, \sum_{g \in G} \omega^i(\xi, g) \leq W \).

**Assumption [A3].** The depreciation structure is given by: \( [Y(\xi)] = [\text{diag}[a(\xi, g)]]_{g \in G} \) and there exists \( k \in (0, 1) \) such that for each node \( \xi \in D, \max_{g \in G} \{a(\xi, g)\} \leq k \).

**Assumption [A4].** For each agent \( i \in I \), for each node \( \xi \in D \) and for each asset \( j \in J(\xi) \), we assume that:

(i) The function \( F^{i,j}_\xi \) is continuous and non-increasing\(^2\) on \( \mathbb{R}_{++}^{(\xi^-)} \).

(ii) The function \( F^{i,j}_\xi \) is concave on \( \mathbb{R}_{++}^{(\xi^-)} \).

(iii) There exists \( \Delta^i(\xi) > 0 \) such that \( F^{i,j}_\xi(\Delta^i(\xi)) > 0 \), for all \( \Delta^i(\xi) < \Delta^i(\xi) \).

Assumptions [A1], [A2] and [A3] are classical in infinite horizon models with default and collateral requirement (see Araujo et al. (2002) for instance). Item (i) of Assumption [A4] is required to ensure that agents’ maximization problems have a solution. For a finite–horizon economy with a continuum of agents, Sabarwal (2003) made assumptions of continuity and non-increasing in defaults similar to [A4 (i)]. The concavity assumed in [A4 (ii)] states that diversified defaults are more tolerated than extreme defaults. It guarantees the convexity of agents’ budget sets. Finally, Item (iii) assumes that default is tolerable up to a certain default level beyond which the defaulter may be excluded from the credit market. This assumption guarantees the nonemptiness of the interior of the individual

\(^1\) For each \( x, y \in \mathbb{R}_{++}^G, y > x \implies v_i^G(y) > v_i^G(x) \).

\(^2\) For each \( \Delta, \Delta' \in \mathbb{R}_{++}^{(\xi^-)}, \Delta \geq \Delta' \implies F^{i,j}_\xi(\Delta) \leq F^{i,j}_\xi(\Delta') \).
budget set (see Claim 2 in Appendix). For the same purpose, Sabarwal (2003) assumes that initial endowments are bounded from below by (exogenous) exemptions. However, the introduction of a structure of exemptions in Sabarwal (2003) has led to non-convexity of the model.

**Remark 3.** In contrast to Zhang (1997) and Alvarez and Jermann (2000), the credit constraints used in this paper do not necessarily exclude all defaulters from the credit markets. Indeed, such default penalties, which actually would make our model similar to one with no default, would not be continuous and, therefore, Item (i) of Assumption [A4] would be violated. However, our model does not rule out the possibility of excluding borrowers from the financial market if their default exceeds the default level $\Delta^i(\xi)$.

4. Generating Ponzi schemes in the presence of credit constraints.

4.1. Opportunity of doing Ponzi schemes. This section illustrates how Ponzi schemes may arise in infinite-horizon with collateral requirement when credit constraints penalizing default are introduced. More precisely, it shows that when the credit constraints are severe, an agent with monotone preferences can always improve upon any budget feasible plan by changing his default and/or his short-sales. Therefore, his maximization problem may have no solution.

In a model with collateral requirement and utility penalties, Pascoa and Seghir (200) have illustrated the occurrence of Ponzi schemes by increasing short-sales at all successors of some node $\xi$ (including node $\xi$) and decreasing default at all strict successors of $\xi$. In this model, increasing short-sales at node $\xi$ may require a simultaneous decrease (at the same node $\xi$) of default on assets sold at node $\xi^-$ (as the value of what an agent is allowed to borrow is constrained according to the amount of his default). More precisely, when the credit constraints penalizing default are not binding (i.e.: what the agent is actually short-selling is less than what he is allowed to short-sell given his default level), an agent can increase his short-sales without changing his default level (see Case 1 below). However, when the short-sales constraints are binding (i.e.: what an agent is actually short-selling is equal to what he is allowed to short-sell given his default level), an agent has to decrease his default level in order to borrow more (see Case 2 below). We prove hereafter that, in either case, an agent can always improve upon any budget feasible plan by adjusting his default and his short-sales if the value of collateral requirements is persistently lower than the loan value.

Formally, let $(p, q, K)$ be a system of prices and expected delivery rates and let $(x^i, \theta^i, \varphi^i, \Delta^i)$ be a collection of individual choice variables of an agent $i \in I$ such that $(x^i, \theta^i, \varphi^i, \Delta^i) \in B^i(p, q, K)$. Let us fix a node $\xi \in D$ and define the following set:

$$\Theta^i(\xi) := \left\{ \sigma \in D(\xi) : \forall j \in J(\sigma), \varphi^i_j(\sigma) = F^{i,j}_\sigma(\Delta^i(\sigma)) \right\}.$$
Assume that:

\[ (5) \quad \forall \sigma \in D(\xi), \exists j \in J(\sigma) : p(\sigma)C^j(\sigma) - q_j(\sigma) < 0. \]

Inequality (5) requires the joint operation of constituting collateral and short-selling the asset to have a negative net price. In other words, it requires loans to exceed collateral cost. Let us distinguish the two following cases:

- **Case 1**: \( \Theta^i(\xi) = \emptyset \), i.e.: \( \forall \sigma \in D(\xi), \exists j_\sigma \in J(\sigma) : \varphi^i_{j_\sigma} < F^i_{j_\sigma}(\Delta^i(\sigma)) \). In other words, at each successor of node \( \xi \), there exists an asset for which the credit constraints penalizing default are not binding. In such a case, let us consider the following changes on agent \( i \)'s short-sales from node \( \xi \) onwards:

\[ (6) \quad \forall \sigma \in D(\xi), \forall j \in J(\sigma), \tilde{\varphi}^i_j(\sigma) = \begin{cases} 
\varphi^i_{j_\sigma}(\sigma) + \varepsilon_\sigma & \text{if } j = j_\sigma \\
\varphi^i_j(\sigma) & \text{if } j \neq j_\sigma
\end{cases}, \varepsilon_\sigma > 0 \]

for some \( j_\sigma \in J(\sigma) \) such that \( \varphi^i_{j_\sigma} < F^i_{j_\sigma}(\Delta^i(\sigma)) \). That is, for each successor \( \sigma \) of \( \xi \), the short-sales on assets \( j_\sigma \), for which the short-sales constraints are not binding, are increased by \( \varepsilon_\sigma \). Note that when short-sales constraints are not binding, short-sales may be increased without decreasing default.

Since the credit constraints are not binding, one can choose \( \varepsilon_\sigma > 0 \), small enough, such that \( (\tilde{x}^i, \tilde{\theta}^i, \tilde{\varphi}^i, \Delta^i) \) satisfies the credit constraints penalizing default (4). Moreover, in view of Inequality (5), it is easy to verify that \( (x^i, \theta^i, \varphi^i, \Delta^i) \) satisfies the budget constraints (2) and (3).

- **Case 2**: \( \Theta^i(\xi) \neq \emptyset \), i.e.: \( \exists \sigma \in D(\xi) : \forall j \in J(\sigma), \varphi^i_j(\sigma) = F^i_{\sigma,j}(\Delta^i(\sigma)) \). In other words, there is some successor of node \( \xi \) for which the credit constraints penalizing default are binding for all assets.

In this case, Ponzi schemes may be generated by decreasing default and increasing short-sales at nodes \( \sigma \in \Theta(\xi) \) for some asset \( j \in J(\sigma^-) \) for which \( \Delta^i_j(\sigma) \neq 0 \). Formally, let us consider the following changes on default and short-sales from node \( \xi \) onwards:

\[ (7) \quad \forall \sigma \in D(\xi), \Delta^i_j(\sigma) = \begin{cases} 
\Delta^i_j(\sigma) & \text{if } \sigma \notin \Theta(\xi) \\
\Delta^i_j(\sigma) - \alpha_\sigma & \text{if } \sigma \in \Theta(\xi)
\end{cases}, \alpha_\sigma > 0. \]

\[ (8) \quad \forall \sigma \in D(\xi), \tilde{\varphi}^i_j(\sigma) = \begin{cases} 
\varphi^i_j(\sigma) & \text{if } \sigma \notin \Theta(\xi) \\
\varphi^i_j(\sigma) + \varepsilon_\sigma & \text{if } \sigma \in \Theta(\xi)
\end{cases}, \varepsilon_\sigma > 0. \]
For each \( \sigma \in \Theta(\xi) \), when \( F^{i,j}_{\sigma} \) is decreasing, one gets:
\[
F^{i,j}_{\sigma}(\Delta^i(\sigma)) = F^{i,j}_{\sigma}\left(\Delta^i(\sigma) - (0, \ldots, 0, \alpha_\sigma, 0, \ldots, 0)\right) > F^{i,j}_{\sigma}(\Delta^i(\sigma)) = \varphi^i_{j,\sigma}(\sigma).
\]
Thus, one can find \( \varepsilon_\sigma > 0 \) such that \( F^{i,j}_{\sigma}(\Delta^i(\sigma)) \geq \varphi^i_{j,\sigma}(\sigma) + \varepsilon_\sigma = \tilde{\varphi}^i_{j,\sigma}(\sigma) \). That is, \( \varepsilon_\sigma \) and \( \alpha_\sigma \) can be chosen such that \((\tilde{\varphi}^i, \tilde{\Delta}^i)\) satisfies the credit constraint (4).
In view of Inequality (5), \((x^i, \theta^i, \tilde{\varphi}^i, \tilde{\Delta}^i)\) satisfies the budget constraints (2) and (3) at nodes \( \sigma \in \Theta(\xi) \) if the following inequality holds at nodes \( \sigma \in \Theta(\xi) \):
\[
\left(p(\sigma)C^i(\sigma) - q^i(\sigma)\right) \varepsilon_\sigma \leq -\alpha_\sigma.
\]
Note that Inequality (9) is satisfied if \( \varepsilon_\sigma \) can be chosen large enough relatively to \( \alpha_\sigma \).

In other words, agent \( i \) may end up doing Ponzi schemes if the credit constraints penalizing default are such that when default on some asset decreases, while default on the other assets is kept constant, the credit opportunities increase at a higher rate than the default’s drop. Therefore, when the net price of the joint operation of constituting the required collateral and short-selling the asset satisfies the sufficient conditions (5) and (9) for occurrence of Ponzi schemes, then agent \( i \) could increase his consumption at node \( \xi \) by decreasing his default and increasing his borrowed values by a higher rate. In such a case, the maximization problem of an agent, with monotone preferences, has no solution and therefore, equilibrium may fail to exist.

The Ponzi scheme illustrated by (7) and (8) requires that agents can simultaneously decrease their default level and increase their short-sales by a higher rate. This brings about the following definition:

**Definition 4.** [Elasticity of credit constraints]

Let \( i \in I, \xi \in D, j \in J(\xi), k \in J(\xi^-) \) and \( \varepsilon > 0 \). We define the elasticity, \( E_{k,\xi}^i F_{\xi}^{i,j}(\delta) \), of the credit constraint function \( F_{\xi}^{i,j} \) at a default level \( \delta \in \mathbb{R}^+_+^{J(\xi^-)} \) as follows:
\[
E_{k,\xi}^i F_{\xi}^{i,j}(\delta) = \frac{F_{\xi}^{i,j}(\delta + \varepsilon \mathbb{I}_k) - F_{\xi}^{i,j}(\delta)}{\varepsilon} \cdot \frac{\delta \cdot \mathbb{I}_k}{F_{\xi}^{i,j}(\delta)},
\]
where \( \mathbb{I}_k = (0, \ldots, 0, 1, 0, \ldots, 0) \) denotes a vector of \( \mathbb{R}^+_+^{J(\xi^-)} \), whose components are equal to zero except the \( k \)th component which is equal to 1.

Moreover, the credit constraints defined by \( F_{\xi}^{i,j} \) are said to be:

(i) inelastic at \( \delta \in \mathbb{R}^+_+^{J(\xi^-)} \) if \( E_{k,\xi}^i F_{\xi}^{i,j}(\delta) \geq -1 \), for any \( \varepsilon > 0 \) and for all \( k \in J(\xi^-) \), i.e.: as default on any asset \( k \in J(\xi^-) \) increases, while keeping default on the other assets constant, the credit opportunities decrease at a lower rate.

(ii) elastic at \( \delta \in \mathbb{R}^+_+^{J(\xi^-)} \) if \( E_{k,\xi}^i F_{\xi}^{i,j}(\delta) < -1 \), for any \( \varepsilon > 0 \) and for all \( k \in J(\xi^-) \) i.e.: as default on any asset \( k \in J(\xi^-) \) increases, while keeping default on the other assets constant, the credit opportunities decrease at a higher rate.
(iii) perfectly elastic at $\delta \in \mathbb{R}_+^{J(\xi^-)}$ if $E^k_\xi F^{i,j}_\xi(\delta)$ is infinite, for any $\varepsilon > 0$ and for all $k$ in $J(\xi^-)$ i.e.: defaulter $i$ is excluded from the financial market independently of the default amount.

(iv) perfectly inelastic at $\delta \in \mathbb{R}_+^{J(\xi^-)}$ if $E^k_\xi F^{i,j}_\xi(\delta) = 0$, for any $\varepsilon > 0$ and for all $k$ in $J(\xi^-)$ i.e.: increasing default does not affect credit opportunities.

**Remark 4.** Araujo et al. (2002) consider a model where the unique punishment in case of default is the seizure of the constituted collateral. Such a punishment structure can be seen as perfectly inelastic for all default levels. In such a model, borrowers pay exactly the minimum between the value of the depreciated collateral and the value of the debt. Therefore, the returns from the joint operation of borrowing and securing the short–sale are always non–negative. By non–arbitrage, it follows that the borrowed value has to be less than or equal to the collateral cost, implying that Condition (5) does not hold.

However, when an additional active enforcement mechanism, such as credit constraints, is introduced, collateral cost may exceed or may be lower than loans and, therefore, Condition (5) may hold. In addition, if the structure of credit constraints is elastic, Inequality (9) may be satisfied which is sufficient for the occurrence of Ponzi schemes.

Ponzi schemes may also occur when the credit constraints are perfectly elastic. Zhang (1997) studies a model in which debt contracts are enforced by exclusion from financial markets. Under such perfectly elastic penalties, the author imposes exogenous short-sale constraints to rule out Ponzi schemes. Finally, Braido (2008) considers a model with numeraire incomplete markets and credit constraints penalizing default. In order to obtain a stationary Markovian equilibrium, the author assumes that the credit constraints are uniformly bounded along the event-tree. Such a requirement (exogenously) rules out Ponzi schemes.

5. Flexibility and equilibrium existence.

5.1. Flexibility. As mentioned in Remark 4, Ponzi schemes can be generated in the presence of elastic (or perfectly elastic) credit constraints. In this section, an assumption regarding the elasticity of the credit constraint functions is introduced to rule out the sufficient conditions of occurrence of Ponzi schemes.

**Assumption [A5].** For each node $\xi \in D$, for each agent $i \in I$, there exists a date $t_{(i)} > t(\xi)$ such that $\forall \sigma \in D^+(\xi) : t(\sigma) = t_{(i)}$, for each asset $j \in J(\sigma)$ one has:

$$
F^{i,j}_{\sigma}(\delta + \varepsilon \mathbb{1}_\kappa) \geq F^{i,j}_{\sigma}(\delta) - \varepsilon \frac{F^{i,j}_{\sigma}(\delta)}{\delta} \mathbb{1}_\kappa,
$$

for each $\varepsilon > 0$, for each asset $\kappa \in J(\sigma^-)$ and for each default level $\delta \in \mathbb{R}_+^{J(\xi^-)}$ such that $\delta \leq \|A^j(\xi)\|_1 W^j T_{(1-k)\sigma(\xi^-)}$, where $c^j(\xi^-) = \min\{C^j_\delta(\xi^-) : C^j_\delta(\xi^-) > 0\}$. 

Assumption [A5] requires that at each date, there is some date in the future in which credit constraints are inelastic. As emphasized by Pascoa and Seghir (2009), agents may end up doing Ponzi schemes, if effective payments are greater than the minimum between the value of the debt and the value of the depreciated collateral, always in the future. Assumption [A5] guarantees that agents will not deliver more than the collateral, sometime in the future and, consequently it ensures that the sufficient conditions (5) and (9) for the occurrence of Ponzi schemes do not hold.

5.2. Equilibrium Existence Result. The main existence result is stated below:

Theorem 1. Under assumptions [A1], [A2], [A3], [A4] and [A5], the economy $E$ has a non–trivial equilibrium.

Proof of Theorem 1. See Appendix.

Remark 5. (On the Transversality conditions). In order to prevent Ponzi schemes in a model with no default, Magill and Quinzii (1994) assume the following transversality condition:

$\lim_{T \to +\infty} \sum_{\xi' \in D(\xi)} \pi^i(\xi') q(\xi') \left( \theta(\xi') - \varphi(\xi') \right) = 0,$

where $\pi^i(\xi')$ is a personalized discount factor. Condition (11) states than agent $i$ is neither a borrower nor a lender at infinity.

When default is allowed, Araujo et al. (2002) show that Condition (11) is not necessarily satisfied even though an equilibrium exists. However, Araujo et al. (2002) prove that, when collateral is the unique enforcement mechanism, the following condition will be endogenously satisfied:

$\lim_{T \to +\infty} \sum_{\{\xi' \in D(\xi) : t(\xi') = T\}} \pi^i(\xi') \left( p(\xi') C(\xi') - q(\xi') \right) \varphi^i(\xi') + \lim_{T \to +\infty} \sum_{\{\xi' \in D(\xi) : t(\xi') = T\}} \pi^i(\xi') q(\xi') \theta^i(\xi') \leq 0.$

Since the no-arbitrage condition in Araujo et al. (2002) requires borrowed values to be less than the value of the constituted collateral, i.e.: $(p(\xi') C(\xi') - q(\xi')) > 0$, it follows from Inequality (12) that:

$\lim_{T \to +\infty} \sum_{\{\xi' \in D(\xi) : t(\xi') = T\}} \pi^i(\xi') \left( p(\xi') C(\xi') - q(\xi') \right) \varphi^i(\xi') = \lim_{T \to +\infty} \sum_{\{\xi' \in D(\xi) : t(\xi') = T\}} \pi^i(\xi') q(\xi') \theta^i(\xi') = 0.$

When collateral seizure and credit constraints penalizing default coexist, the no-arbitrage condition of Araujo et al. (2002) no longer holds. That is, the discounted value of the net cost of borrowing does not necessarily tend to zero as $T$ goes to infinity, Thus, Condition (13) does not necessarily hold and, therefore, a transversality condition similar to (11) needs to be introduced in order to
prevent Ponzi schemes. However, when Assumption [A5] holds, collateral cost must be larger than
loans implying that Condition (13) is satisfied and that Ponzi schemes are ruled out.

5.3. Example. The following example shows that if the credit constraints are elastic, agents end
up doing Ponzi schemes.

Let us consider a model satisfying:

\[(14) \forall \xi \in D, \forall j \in J(\xi), \forall \eta \in \xi^+, A^j(\eta) > Y(\eta) C^j(\eta).\]

Let us consider an agent \(i\) in \(I\). For each \(\xi \in D\) and \(j \in J(\xi)\), let us define the function \(F_{i,j}^{\xi}(\delta)\) by:

\[F_{i,j}^{\xi}(\delta) = \frac{\delta}{\|A^j(\xi)\|_1} + \frac{W I}{(1-k)\sigma^j(\xi)}.\]

It can be easily shown that \(F_{i,j}^{\xi}\) is elastic for all \(\delta\) such that \(\frac{\|A^j(\xi)\|_1 W I}{2(1-k)\sigma^j(\xi)} \leq \delta \leq \frac{\|A^j(\xi)\|_1 W I}{(1-k)\sigma^j(\xi)}\). That is, Assumption [A5] is violated in this example. Moreover, in view of Condition (14) and the elasticity
of \(F_{i,j}^{\xi}\), the net returns \((pYC - pA)\varphi\) from the joint operating of short selling the security and
buying the required collateral are negative. By a non-arbitrage argument, the net price, \((pC - q)\),
must be negative as well. Therefore, Condition (5) is satisfied.

Let \((p, q, K)\) such that \(q(\xi) \neq 0, \forall \xi \in D\). Let us consider an allocation \((x^i, \varphi^i, \theta^i, \Delta^i) \in B^i(p, q, K)\).

Let us consider the following changes on agent \(i\)'s default and short–sales: \(\Delta^i(\xi) = \Delta^i(\xi) - \varepsilon\) and
\(\varphi^i(\xi) = \varphi^i(\xi) + \alpha\). In view of Condition (5) and the elasticity of \(F_{i,j}^{\xi}\), one gets that \((x^i, \varphi^i, \theta^i, \Delta^i)\)
satisfies conditions (2), (3) and (4). Moreover, one can choose \(\varepsilon\) large enough relatively to \(\alpha\) so
that Condition (9) is satisfied. Thus, the sufficient conditions, (5) and (9), for occurrence of Ponzi
schemes are satisfied in this example.

Appendix.

Proof of Theorem 1. To show the existence of a nontrivial equilibrium, one can prove that in
equilibrium, commodities prices are bounded from below (using the monotonicity of preferences).
Then, since asset payments are greater than or equal to the minimum of the promise and the
depreciated collateral, one can get a lower bound for unitary payments of borrowers. This lower
bound on payments induces a lower bound on delivery rates. Thus, one can insert, in the abstract
economy, a lower bound on payments and prove that this lower bound, if properly chosen, is not
binding (see Steinert and Torres-Martinez (2007) and Pascoa and Seghir (2009) for more details).
The proof of Theorem 1 is done in two main steps. The first step shows the equilibrium existence
in truncated economies while the second step is devoted to asymptotic results.

Step 1: Equilibria in truncated economies.

Let \(E^T\) be the truncated economy associated with the original economy \(E\), which has the same
characteristics as \(E\), but where we suppose that agents are constrained to stop their exchange of
goods at period $T$ and their trade of assets at period $T - 1$. Formally, for each $T > 0$, let us define the following sets:

$$\Pi^{T-1} := \left\{ (p, q) \in \mathbb{R}_+^{D \times G} \times \prod_{\xi \in D^T} \mathbb{R}^{(\xi)} \middle| \forall \xi : t(\xi) < T, \|p(\xi)\|_1 + \|q(\xi)\|_1 = 1, \right.$$  
$$\forall \xi : t(\xi) = T, \|p(\xi)\|_1 = 1. \right\},$$

$$K^T := [0, 1]^{(\sum_{\xi \in D^T} \ell(\xi))},$$

and for each $i \in I$,

$$X^{iT} = \{(x^i(\xi), \xi \in D) \in X^i \mid \forall \xi : t(\xi) > T, x^i(\xi) = 0\},$$

$$Z^{iT} = \{(z^i(\xi), \xi \in D) \in X^i \mid \forall \xi : t(\xi) > T, \theta^i(\xi) = \varphi^i(\xi) = 0\}.$$

Moreover, given $(p, q, K) \in \Pi^{T-1} \times K^T$, the budget set, $B^{iT}(p, q, K)$, of an agent $i \in I$ for the truncated economy is defined by the set of $(x, z, \Delta)$ such that $x^i \in X^{iT}$, $z^i \in Z^{iT}$, (2) holds at $\xi = 0$ and (3)-(4) hold at all the other nodes. In addition, for each agent $i \in I$, the utility function $U^{iT}$ for each truncated economy $E^T$ is defined as follows: $U^{iT}(x^i, \theta^i, \varphi^i, \Delta^i) := \sum_{\xi \in D^T} v^i_\xi(\tilde{x}^i(\xi))$.

**Definition 5.** [Equilibria of the truncated economies]

An equilibrium of $E^T$ is a collection $(p^T, \theta^T, \varphi^T, \Delta^T, (p^{iT}, \theta^{iT}, \varphi^{iT}, \Delta^{iT})_{i \in I})$ verifying:

(a) For each agent $i \in I$, $(p^T, \theta^T, \varphi^T, \Delta^T) \in \text{Argmax } U^{iT}(x)$ over $B^{iT}(p^T, \theta^T, \varphi^T, \Delta^T)$,

(b) Conditions (ii)–(v) of Definition 2 hold at $(p^T, \theta^T, \varphi^T, \Delta^T)$ for $\xi \in D^T$, with $\varphi^T(\xi) = 0$ when $t(\xi) = T$.

An equilibrium of $E^T$ is said to be non–trivial if it satisfies the following condition:

(c) For any $(\xi, j)$, either $\left(\theta_j(\xi), \varphi_j(\xi)\right)$ is different from 0 or $\varphi_j(\xi) > 0$.

**Proposition 1.** Under assumptions [A1] and [A2], each truncated economy $E^T$ has a non–trivial equilibrium $(p^T, \theta^T, \varphi^T, \Delta^T, (p^{iT}, \theta^{iT}, \varphi^{iT}, \Delta^{iT})_{i \in I})$.

**Proof.** The proof of Proposition 1 is analogous to the proof of the existence of a non–trivial equilibrium in Pascoa and Seghir (2006). However, there is a dissimilarity as for the non-emptiness of the equilibrium as the similarities with the proof in Pascoa and Seghir (2006) are substantial.
interior of the budget set. In fact, as it will be explained hereafter, the credit constraint penalizing default (4) and the alteration of the decision variables in this model compared to Pascoa and Seghir (2006) makes the non–emptiness of the interior of the budget sets more problematic. To prove the non-emptiness of the interior of the budget sets, Pascoa and Seghir (2009) set short-sales equal to zero at the first period and effective payment strictly positive at the following period. Due to the presence of credit constraint functions in our model and since these functions may have negative values for some default levels, the idea used in Pascoa and Seghir (2009) would no longer hold (see Claim 2 below).

Lemma 1. Under Assumption [A2], an allocation \((x, \theta, \varphi, \Delta)\) which satisfies the conditions of Definition 5 is bounded.

Proof. Following the same idea as in Araujo et al. (2002), one gets:

\[
x^i(\xi, g) \leq W T \sum_{n=0}^{t} (\bar{Y}_T g)^n := \zeta^T < +\infty, \ \forall g \in G,
\]

\[
\varphi^j_i(\xi) \leq \frac{\zeta^T}{\eta^j(\xi)} := \alpha^T(\xi) < +\infty, \ \forall j \in J(\xi),
\]

\[
\theta^j_i(\xi) \leq \alpha^T(\xi) < +\infty, \ \forall j \in J(\xi),
\]

where \(\bar{Y}_T := \max\{\{(Y(\xi))_{g,g'}(\xi, g, g') \in D_T \times G \times G\} \text{ and } \eta^j(\xi) = \min\{C^j_g(\xi) : C^j_g(\xi) > 0\}\).

For each node \(\xi \in D^T\), let us define \(\chi^T(\xi) = \max\{\zeta^T(\xi), \alpha^T(\xi)\}\) and \(\chi^T = \max_{\xi \in D_T} \chi^T(\xi)\). Now, for each \(i \in I\), let us define:

\[
B^iT(p, q, K, \chi) = \left\{(x, \theta, \varphi, \Delta) \in B^iT(p, q, K) \left| \begin{array}{c} x^i(\xi, g) \leq 2\chi^T, \\
\theta^j_i(\xi) \leq 2\chi^T, \\
\varphi^j_i(\xi) \leq 2\chi^T, \\
\end{array} \right. \right\}
\]

Let \(\mathcal{E}_T(\chi)\) be the compactified economy which has the same characteristics as \(\mathcal{E}_T\) except for the budget constraints which are now defined by the sets \(B^iT(p, q, K, \chi)\).

Definition 6. An equilibrium of the compactified economy \(\mathcal{E}_T(\chi)\) is a vector

\[
(p_T, q_T, \bar{K}_T, (\bar{\theta}_T, \bar{\varphi}_T, \bar{\Delta}_T)_{i \in I})
\]

verifying conditions (b) and (c) of Definition 5 and such that:

\[(i') \ \forall i \in I, \ (\bar{\theta}_T, \bar{\varphi}_T, \bar{\Delta}_T)_{i \in I} \in \text{Argmax } U^iT(x) \text{ over } B^iT(p, q, K, \chi)\).

Lemma 2. Under Assumptions [A1] and [A2], each compactified economy \(\mathcal{E}_T(\chi)\) has a non–trivial equilibrium \((p_T, q_T, \bar{K}_T, (\bar{\theta}_T, \bar{\varphi}_T, \bar{\Delta}_T)_{i \in I})\).
Proof. Note that for each \( i \in I, B^{IT}_{\xi} \) is upper semicontinuous with nonempty closed convex values. For each \((p,q,K) \in \Pi^{T-1} \times [0,1]^{\bigcup_{i \in D^T} J(i)}\) and each agent \( i \in I, \) let us define the set \( B^{\alpha T}_{\xi}(p,q,K,\chi) \) by replacing all the inequalities in \( B^{IT}_{\xi}(p,q,K,\chi) \) by strict inequalities.

**Claim 2.** \( \forall i \in I, \forall (p,q,K) \in \Pi^{T-1} \times [0,1]^{\bigcup_{i \in D^T} J(i)}, B^{\alpha T}_{\xi}(p,q,K,\chi) \neq 0. \)

**Proof.** The proof is done by upward induction as follows:

- At node \( \xi = \xi_0, \)
  - If \( p(\xi_0) \neq 0, \) since \( \omega^i(\xi_0) \gg 0, \) one can choose \( x^i(\xi_0) \gg 0, \) \( \varphi^i(\xi_0) > 0, \) \( \varphi^i(\xi_0) \) small enough, such that \( p(\xi_0) \cdot \left[ x^i(\xi_0) + C(\xi_0) \varphi^i(\xi_0) \right] < p(\xi_0) \cdot \omega^i(\xi_0). \) Letting \( \theta^i(\xi_0) = 0, \) one gets that the constraints of the period 0 are satisfied strictly.
  - If \( p(\xi_0) = 0 \) (then \( q(\xi_0) \neq 0, \) one can choose \( \theta^i(\xi_0) = 0, \) \( \varphi^i(\xi_0) \gg 0, \) \( \varphi^i(\xi_0) \) small enough, such that \( q_j(\xi_0) \cdot \varphi^i(\xi_0) > 0 \) and the constraints of the period 0 will be satisfied strictly.

- At each \( \xi \in \xi_0^+, \)
  - If \( p(\xi) \neq 0, \) since \( [\omega^i(\xi) + Y(\xi) x^i(\xi_0)] \gg 0, \) one can choose \( x^i(\xi) \gg 0, \) \( \varphi^i(\xi) > 0, \) \( \varphi^i(\xi) \) small enough and \( \Delta^i(\xi) < \tilde{\Delta}^i(\xi) \) such that
    \[
    p(\xi) \cdot \left[ x^i(\xi) + C(\xi) \varphi^i(\xi) \right] < p(\xi) Y(\xi) \left[ x^i(\xi_0) + C(\xi_0) \varphi^i(\xi_0) \right]
    \]
    and
    \[
    \varphi^i_j(\xi) < F_{\xi}^{\alpha j} \left( \Delta^i(\xi) \right).
    \]
    Note that \( \varphi^i(\xi) \) can be chosen small enough to satisfy Inequality (18) and this \( \varphi^i(\xi) \) is compatible with the default \( \Delta^i \) satisfying (19). Letting \( \theta^i(\xi) = 0, \) one gets that the constraints of node \( \xi \) are satisfied strictly.
  - If \( p(\xi) = 0 \) (then \( q(\xi) \neq 0, \) one can choose \( \varphi^i(\xi) > 0 \) and \( \Delta^i(\xi) > 0 \) such that 
    \( \Delta^i(\xi) < \tilde{\Delta}^i(\xi) \) and \( \varphi^i_j(\xi) < F_{\xi}^{\alpha j} \left( \Delta^i(\xi) \right). \)

- The same ideas can be used until the period \( T - 1. \)
- At node \( \xi \in D_T \) (i.e. \( t(\xi) = T \)). Since \( p(\xi) \neq 0, \) one can choose \( x^i(\xi) \in X^i(\xi) \) such that \( p(\xi) \cdot x^i(\xi) < p(\xi) \cdot [\omega^i(\xi) + Y(\xi) x^i(\xi^-)] \).

\( \square \)

**Claim 3.** \( \forall i \in I, B^{IT}_{\xi} \) is lower semicontinuous.

**Proof.** It follows from the convexity and the non-emptiness of \( B^{\alpha T}_{\xi}(p,q,K,\chi) \) for each \((p,q,K) \in \Pi^{T} \times [\frac{1}{n},1]^{D^T \times \bigcup_{i \in D^T} J(i)}\) that \( B^{IT}_{\xi}(p,q,K,\chi) = B^{\alpha T}_{\xi}(p,q,K,\chi). \) The Claim follows from the fact that
$B^{iT}$ is lower semicontinuous

The end of the proof of Proposition 1 follows the same arguments as in Pascoa and Seghir (2009) using Kakutani fixed point theorem and convexity arguments.

**Step 2: Asymptotic Results.**

Under assumptions $[A1]$, $[A2]$ and $[A3]$, one has for each node $\xi \in D^T$:

$$\sum_{(i,g) \in I \times G} \frac{1}{\sigma_i(\xi)} \frac{W \mathcal{I}}{1-k} \leq 1,$$

$$\sum_{i \in I} \xi_j^i(\xi) \leq \frac{1}{\sigma_j(\xi)} \frac{W \mathcal{I}}{1-k},$$

where $\sigma_j(\xi) = \min\{C_j^i(\xi) : C_j^i(\xi) > 0\}$. In view of conditions (20)–(23) and the countability of $D$, we get, via a diagonalization procedure as in Araujo et al. (2002), a sequence $(\tilde{T}_k, \tilde{\theta}_k, \tilde{\varphi}_k, \tilde{\varpi}_k, \tilde{\Delta}_k)$ which converges, at each node, to some $((\pi, \bar{\theta}, \bar{\varphi}, \bar{\Delta}), \bar{p}, \bar{q}, \bar{K})$.

**Proposition 2.** For each agent $i \in I$, the cluster point $(\pi^i, \bar{\theta}^i, \bar{\varphi}^i, \bar{\Delta}^i)$ is optimal in $B^i(\bar{p}, \bar{q}, \bar{K})$.

**Proof.** Suppose that there exists an agent $i$ and a plan $(\tilde{x}^i, \tilde{\theta}^i, \tilde{\varphi}^i, \tilde{\Delta}^i) \in B^i(\bar{p}, \bar{q}, \bar{K})$ such that:

$$U^i(\tilde{x}^i) = U^i(\pi^i) > 0.$$ Then, there is $\mathcal{T}$ such that for every $T > \mathcal{T}$, $\sum_{\xi \in D^T} \psi^i(\tilde{x}^i) > U^i(\pi^i)$.

Let us fix $\tilde{T} > \mathcal{T}$ such that for each node $\sigma : t(\sigma) = \tilde{T} + 1$ the credit constraint functions satisfy Equation (10) of Assumption $[A5]$. For each plan $y := (y(\xi), \xi \in D)$ let us define the following correspondence: $\psi^\tilde{T}$ and $\beta^\tilde{T}$:

$$\psi^\tilde{T}(y) := \left\{(x(\xi), \xi \in D^\tilde{T}) \mid U^{iT}(x) > U^i(y)\right\}.$$

Moreover, for each price and expected delivery rate process $(p, q, K)$, let us define the following correspondence:

$$\beta^\tilde{T}(p, q, K) = \{(x(\xi), \xi \in D^\tilde{T}) : \exists (\theta(\xi), \varphi(\xi), \Delta(\xi), \xi \in D^\tilde{T}) \text{ s. t. } (x(\xi), \theta(\xi), \varphi(\xi), \Delta(\xi)) \text{ satisfies the budget constraints (3), (4) and (5)} \}.$$
Since \( \hat{x} \in \beta^T(p, q, K) \cap \psi^T(\pi) \) and \( \beta^T(p, q, K) \cap \psi^T(\pi) \) is lower semicontinuous with respect to the product topology on \( L^\infty(D) \) (recall that \( U^i \) is weak star upper semicontinuous and apply Hildenbrand (1974), p.35, Prob.6 (1)), one gets the existence of \( T^* \) and a sequence \( \hat{x}^T \) converging, node by node, to \( \hat{x} \) such that \( \forall T \geq T^* \), \( \hat{x}^T \in \beta^T(p^T, q^T, K^T) \cap \psi^T(\pi^T) \).

With no loss of generality, one can assume that \( T^* > \tilde{T}. \) Take \( T = T^* \) to get that \( U^{iT}(\hat{x}^T) > U^{jT}(\pi^T) \) and the existence of (\( \hat{\theta}^T, \hat{\varphi}^T, \hat{\Delta}^T \)) such that (\( \hat{x}^T, \hat{\theta}^T, \hat{\varphi}^T, \hat{\Delta}^T \)) satisfies the budget constraints till \( \tilde{T} \) at \( (p^T, q^T, K^T) \).

For simplicity of notations, let us define \( h^T_j(\xi) := p^T(\xi)C_j(\xi) - q^T(\xi, j). \)

Let \( \varepsilon > 0 \) and \( \alpha \in [0, 1] \) and let us define the following changes:

\[
x^T_g(\xi) = \begin{cases} 
\tilde{x}^T_g(\xi, g) & \text{if } t(\xi) \leq \tilde{T} - 1 \\
\tilde{x}^T(\xi, g) + \frac{\varepsilon}{p^T(\xi, g)} & \text{if } t(\xi) = \tilde{T} \\
0 & \text{if } t(\xi) > \tilde{T}.
\end{cases}
\]

\[
\varphi^T_j(\xi) = \begin{cases} 
\tilde{\varphi}^T_j(\xi) & \text{if } t(\xi) \leq \tilde{T} - 1 \\
\tilde{\varphi}^T(\xi, g) - \frac{F(\tilde{\Delta}^T_j(\xi))}{\tilde{\Delta}^T_j(\xi)} & \text{if } t(\xi) = \tilde{T} \text{ and } h^T_j(\xi) > 0 \\
\tilde{\varphi}^T(\xi, g) + \frac{F(\tilde{\Delta}^T_j(\xi))}{\tilde{\Delta}^T_j(\xi)} & \text{if } t(\xi) = \tilde{T} \text{ and } h^T_j(\xi) < 0 \\
0 & \text{if } t(\xi) > \tilde{T}.
\end{cases}
\]

Clearly, \((x^T, \theta^T, \varphi^T, \Delta^T)\) satisfies the budget constraints up to time \( \tilde{T} - 1 \) of the truncated economy \( \mathcal{E}^T \). Moreover, in view of Assumption [A5], \((x^T, \theta^T, \varphi^T, \Delta^T)\) satisfies the credit constraint (4) at \( \tilde{T} \) node \( \xi \). In addition, by choosing this new default-short–sales vector, agent \( i \) can increase his consumption, from \( \hat{x}^T \) to \( x^T \), at node \( \xi \), which contradicts the optimality of \((p^T, \varphi^T, K^T)\) in \( B^i(p^T, q^T, K^T) \).

References.


